ABSTRACT

The present doctoral thesis contains the personal contributions of the author in the study of stochastic processes, mainly the Brownian motion, and its applications in other areas of mathematics, such as Complex Analysis, Differential equations and Numerical methods. The theory of stochastic processes is a relatively new area of research in mathematics, its foundations being laid at the beginning of the last century, which had a remarkable development in the last period of time, both theoretically as well as from the point of view of applications – we mention here for example the award in 1997 of the Nobel prize in Economics for the foundation on mathematical grounds of the computation of the stock price options.

The first chapter of the thesis is introductory, having as goal to present the notions and the results needed in the presentation of the results in the next two chapters of the thesis. There are presented here the fundamental notions of probability space, conditional expectation with respect to a σ-algebra, martingale and submartingale/supermartingale, their properties and examples. Next, it is introduced the Brownian motion, stochastic process around which is detailed the analysis in the next chapters. After reviewing the main properties of the Brownian motion, we present some elements of integral calculus with respect to Brownian motion, respectively the Itô stochastic integral. The chapter concludes with the introduction of the reflecting Brownian motion and of the techniques of couplings for such processes, especially the mirror coupling of reflecting Brownian motions, used by the author in proving the results in the last chapter of the thesis.

The main connection between the Brownian motion and the Complex anlysis is Paul Lévy's theorem on the invariance of Brownian motion under composition with analytic functions in the complex plane; under the additional hypothesis that the analytic function is also injective (univalent), the same result being true for the reflecting/killed Brownian motion in a domain. Starting from this important observation, in the second chapter of the thesis the author presents several original results on the conditions which guarantee the univalence of an analytic function, such as the generalization of the Ozaki-Nunokawa univalence criterion, or certain results related to the univalence of certain classical integral operators such as the Kim-Merkes operator or the Pfaltzgraff operator. In this chapter are also presented other original contributions of the author in Complex analysis, such as the extention of the maximum modulus principle and the Schwarz lemma to certain classes of non-analytic functions.

The last chapter of the thesis contains the original contributions of the author in proving certain monotonicity properties of reflecting Brownian motion, and applications related to solving certain differential equations. Remarkable here is the complete resolution of the Laugesen-Morpurgo conjecture on the radial monotonicity of the transition density of the reflecting Brownian motion in the unit disk, and the proof of an integral version of Chavel's conjecture on the domain monotonicity of the transition density of the reflecting Brownian motion. As applications, there are presented the contributions of the author to solving by probabilistic methods certain differential equations, such as the Dirichlet problem for Laplace's equation, or more generally the Dirichlet problem for Poisson's equation. There are also presented here several algorithms for solving these problems, as well as their implementation in the Mathematica program. The thesis concludes with a list of bibliographic references and 4 appendices containing the source codes of the Mathematica programs.