CHAPTER 5: Lie Differentiation and Angular Momentum

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1 Lie differentiation

Kähler's theory of angular momentum is a specialization of his approach to Lie differentiation. We could deal with the former directly, but we do not want to miss this opportunity to show you both, as they are jewels. As an exercise, readers can at each step specialize the Lie theory to rotations.

1.1 Of Lie differentiation and angular momentum

For rotations around the z axis, we have

$$\frac{\partial}{\partial \phi} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}. \tag{1.1}$$

The partial derivative equals an example of what Kähler defines as a Lie operator, i.e.

$$X = \alpha^i(x^1, x^2, ...x^n)\frac{\partial}{\partial x^i}, \tag{1.2}$$

without explicitly resorting to vector fields and their flows (See section 16 of his 1962 paper). Incidentally, $\partial/\partial x^i$ does not respond to the concept of vector field in Cartan and Kähler (For more on these concepts, see section 8.1 of my book "Differential Geometry for Physicists and Mathematicians"). Contrary to what one may read in the literature, not all concepts of vector field are equivalent, but simply related (See section 3.5 of that book).

One would like to make (1.2) into a partial derivative. When I had already written most of this section, I realized that it was not good enough to refer readers to Kähler's 1960 paper in order to know how to do that; until one gets hold of that paper (in German, by the way), many readers would not be able to understand this section. So, we have added the present subsection 1.8 to effect such a change into a partial derivative.

Following Kähler, we write the operator (1.1) as χ_3 since we may extend the concept to any plane. We shall later use

$$\chi_k = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \tag{1.3}$$

where (i, j, k) constitutes any of the three cyclic permutations of $(1, 2, 3)$, including the unity. Here, the coordinates are Cartesian.

Starting with chapter 2 posted in this web site (the first one to be taught in the Kähler calculus phases (II and III) of the summer school), we have not used tangent-valued differential forms, not even tangent vector fields. Let us be more precise. We will encounter expressions that can be viewed as components of vector-valued differential $1-$forms because of the way they transform when changing bases. But those components are extractions from formulas arising in manipulations, without the need to introduce invariant objects of which those expressions may be viewed as components. The not resorting to tangent-valued quantities will remain the case in this chapter, even when dealing with total angular momentum; the three components will be brought together into just one element of the algebra of scalar-valued differential forms.

1.2 Lie operators as partial derivatives

Cartan and Kähler defined Lie operators by (1.2) (in arbitrary coordinate systems!) and applied them to differential forms. A subreptitious difficulty with this operator is that the partial derivatives take place under different conditions as to what is maintained constant for each of them. This has consequences when applied to differential forms.

In subsection 1.8, we reproduce Kähler's derivation of the Lie derivative as a single partial derivative with respect to a coordinate y^n from other coordinate systems,

$$X = \alpha^i(x) \frac{\partial}{\partial x^i} = \frac{\partial}{\partial y^n}. \tag{1.4}$$

His proof of (1.4) makes it obvious why he chose the notation y^n

Let u be a differential form of grade p,

$$u = \frac{1}{p!} a_{i_1 \ldots i_p} dx^{i_1} \wedge \ldots \wedge dx^{i_p}, \tag{1.5}$$

in arbitrary coordinate systems. Exceptionally, summation does not take place over a basis of differential $p-$forms, but over all values of the indices. This notation is momentarily used to help readers connect with formulas in in Kähler's 1960 paper.

Our starting point will be

$$Xa_{i_1...i_p} = \alpha^i(x)\frac{\partial a_{i_1...i_p}}{\partial x^i} = \frac{\partial a_{i_1...i_p}}{\partial y^n}. \tag{1.6}$$

1.3 Non-invariant form of Lie differentiation

In subsection 1.8, we derive

$$Xu = \frac{1}{p!}\alpha^i\frac{\partial a_{i_1...i_p}}{\partial x^i}dx^{i_1}\wedge ... \wedge dx^{i_p} + d\alpha^i \wedge e_i u, \tag{1.7}$$

with the operator e_i as in previous chapters.

Assume that the α^i's were constants. The last term would drop out. Hence, for X_i given by $\partial/\partial x^i$ and for u given by $a_{i_1...i_p}dx^{i_1}\wedge ... \wedge dx^{i_p}$, we have

$$X_i(a_{i_1...i_p}dx^{i_1}\wedge ... \wedge dx^{i_p}) = \frac{\partial(a_{i_1...i_p}dx^{i_1}\wedge ... \wedge dx^{i_p})}{\partial x^i} = \frac{\partial a_{i_1...i_p}}{\partial x^i}dx^{i_1...i_p}, \tag{1.8}$$

where $dx^{i_1...i_p}$ stands for $dx^{i_1}\wedge ... \wedge dx^{i_p}$. This allows us to rewrite (1.7) as

$$Xu = \frac{1}{p!}\alpha^i\left[\frac{\partial(a_{i_1...i_p}dx^{i_1...i_p})}{\partial x^i}\right] + d\alpha^i \wedge e_i u, \tag{1.9}$$

It is then clear that

$$Xu = \alpha^i\frac{\partial u}{\partial x^i} + d\alpha^i \wedge e_i u, \tag{1.10}$$

In 1962, Kähler used (1.10) as starting point for a comprehensive treatment of lie differentiation.

The first term on the right of (1.10) may look as sufficient to represent the action of X on u, and then be overlooked in actual computations. In subsection 1.8, we show that this is not so. We now focus on the first term since it is the one with which one can become confused in actual practice with Lie derivatives.

Notice again that, if the α^i's are constants —and the constants $(0, 0, ..1, 0, ...0)$ in particular— the last term in all these equations vanishes. So, we have

$$X(cu) = c\frac{\partial u}{\partial x^i}, \tag{1.11}$$

for a equal to a constant c. But

$$X\left[a(x)u\right] = a(x)\frac{\partial u}{\partial x^i} \qquad \text{(Wrong!)}$$

is wrong. When in doubt with special cases of Lie differentiations, resort to (1.10).

The terms on the right of equations (1.7) to (1.10) are not invariant under changes of bases. So, if u were the state differential form for a particle, none of these terms could be considered as properties of the particle, say its orbital and spin angular momenta.

1.4 Invariant form of Lie differentiation

Kähler subtracted $\alpha^i \omega_i{}^k \wedge e_k u$ from the first term in (1.10) and simultaneously added it to the second term. Thus he obtained

$$Xu = \alpha^i d_i u + (d\alpha)^i \wedge e_i u, \tag{1.12}$$

since

$$\alpha^i \frac{\partial u}{\partial x^i} - \alpha^i \omega_i{}^k \wedge e_k u = \alpha^i d_i u, \tag{1.13}$$

and where we have defined $(d\alpha)^i$ as

$$(d\alpha)^i \equiv d\alpha^i + \alpha^i \omega_i{}^k. \tag{1.14}$$

One may view $d\alpha^i + \alpha^k \omega_k{}^i$ as the contravariant components of what Cartan and Kähler call the exterior derivative of a vector field of components α^i. By "components as vector", we mean those quantities which contracted with the elements of a field of vector bases yield the said exterior derivative. Both differential-form-valued vector field and vector-field-valued differential $1-$form are legitimate terms for a quantity of that type. The corresponding covariant components are

$$(d\alpha)_i = d\alpha_i - \alpha_h \omega_i{}^h. \tag{1.15}$$

If you do not find (1.15) in the sources from which you learn differential geometry, and much more so if your knowledge of this subject is confined to the tensor calculus, please refer again to my book "Differential Geometry for Physicists and Mathematicians". Of course, if you do not need to know things in such a depth, just believe the step from (1.14) to (1.15). We are using Kähler's notation, or staying very close to it. Nevertheless, there is a more Cartanian way of dealing with the contents of this and the next subsections. See subsection 1.7.

In view of the considerations made in the previous sections, we further have

$$Xu = \alpha^i d_i u + (d\alpha)_i \wedge e^i u \tag{1.16}$$

All three terms in (1.12) and (1.16) are invariant under coordinate transformations. The two terms on the right do not mix when performing a change of basis. This was not the case with the two terms on the right of (1.7) and (1.10), even though their form might induce one to believe otherwise.

1.5 Action of a Lie operator on the metric's coefficients

Following Kähler we introduce the differential 1−form α with components α_i, i.e.

$$\alpha = \alpha_k dx^k = g_{ik}\alpha^i dx^k. \tag{1.17}$$

If the α^i were components of a vector field, the α_k would be its covariant components. But both of them are here components of the differential form α. We define $d_i\alpha_k$ by

$$(d\alpha)_i = (d_i\alpha_k)dx^k. \tag{1.18}$$

Hence, on account of (1.15),

$$d_i\alpha_k \equiv \alpha_{i,k} - \alpha_h \Gamma_i{}^h{}_k. \tag{1.19}$$

Therefore,

$$d_i\alpha_k + d_k\alpha_i = \alpha_{i,k} + \alpha_{k,i} - \alpha_h\Gamma_i{}^h{}_k - \alpha_h\Gamma_k{}^h{}_i \tag{1.20}$$

In a coordinate system where $\alpha^i = 0$ $(i < n)$ and $\alpha^n = 1$, we have

$$\alpha_{i};_{k} = (g_{pi}\alpha^p),_k = g_{pi};_k\,\alpha^p = g_{ni};_k\,, \tag{1.21}$$

and, therefore,

$$\alpha_{i};_{k} + \alpha_{k};_{i} = g_{ni};_{k} + g_{nk};_{i}\,. \tag{1.22}$$

On the other hand,

$$\alpha^l\Gamma_{ilk} + \alpha^l\Gamma_{kli} = 2\Gamma_{ink} = g_{ni,k} + g_{nk,i} - g_{ik,n}, \tag{1.23}$$

From (1.20), (1.22) and (1.23), we obtain

$$d_k\alpha_i + d_i\alpha_k = \frac{\partial g_{ik}}{\partial x^n}. \tag{1.24}$$

1.6 Killing symmetry and the Lie derivative

When the metric does not depend on x^n, (1.24) yields

$$d_k\alpha_i + d_i\alpha_k = 0. \tag{1.25}$$

We then have that

$$e_i d\alpha = -2(d\alpha)_i. \tag{1.26}$$

Indeed,

$$e_i d\alpha = e_i d(\alpha_k dx^k)] = e_i[(\alpha_{k,m} - \alpha_{m,k})(dx^m \wedge dx^k)] = (\alpha_{k,i} - \alpha_{i,k})dx^k, \tag{1.27}$$

where the parenthesis around $dx^m \wedge dx^k$ is meant to signify that we sum over a basis of differential $2-$forms, rather than for all values of i and k. By virtue of (1.18), (1.19) and (1.25), we have

$$2(d\alpha)_i = (d_i\alpha_k - d_k\alpha_i)dx^k = [(\alpha_{i,k} - \alpha_h\Gamma_i{}^h{}_k) - (\alpha_{k,i} - \alpha_h\Gamma_k{}^h{}_i)]dx^k. \tag{1.28}$$

We now use that $\Gamma_i{}^h{}_k = \Gamma_k{}^h{}_i$ in coordinate bases, and, therefore,

$$2(d\alpha)_i = (\alpha_{i,k} - \alpha_{k,i})dx^k = -e_i d\alpha. \tag{1.29}$$

Hence (1.26) follows, and (1.16) becomes

$$Xu = \alpha^i d_i u - \frac{1}{2}e_i d\alpha \wedge e^i u. \tag{1.30}$$

Notice that we have just got Xu in pure terms of differential forms, unlike (1.16), where $(d\alpha)_i$ makes implicit reference to the differentiation of a tensor field.

An easy calculation (See Kähler 1962) yields

$$-2e_i d\alpha = d\alpha \vee u - u \vee d\alpha. \tag{1.31}$$

Hence,

$$Xu = \alpha^i d_i u + \frac{1}{4}d\alpha \vee u - \frac{1}{4}u \wedge d\alpha, \tag{1.32}$$

which is our final expression for the Lie derivative of a differential form if that derivative is associated with a Killing symmetry.

1.7 Remarks for improving the Kähler calculus

The Kähler calculus is a superb calculus, and yet Cartan would have written it without coordinate bases. We saw in chapter one the disadvantage that these bases have relative to the orthonormal ω^i's, which are differential invariants that define a differentiable manifold endowed with a metric. In this section, the disadvantage lies in that one needs to have extreme care when raising and lowering indices, which is not a problem with orthonormal bases since one simply multiplies by one or minus one. Add to that the fact that dx_i does not make sense since there are not such a thing as "covariant curvilinear coordinates". On the other hand, ω_i is well defined.

Consider next the Killing symmetry, (1.25). The $d_k\alpha_i$ are associated with the covariant derivative of a vector field. But they could also be associated with the covariant derivatives of a differential 1-form. Indeed, we define $(d_i\alpha)_k$ by

$$d_i\alpha = (\alpha_{k,i} - \alpha_l\Gamma_k{}^l{}_i)dx^k \equiv (d_i\alpha)_k dx^k. \tag{1.33}$$

But

$$d_k\alpha_i \equiv \alpha_{k,i} - \alpha_h\Gamma_k{}^h{}_i. \tag{1.34}$$

Thus

$$(d_i\alpha)_k = d_k\alpha_i \tag{1.35}$$

and the argument of the previous two sections could have been carried out with covariant derivatives of differential forms without invoking components of vector fields.

1.8 Derivation of Lie differentiation as partial differentiation

Because the treatment of vector fields and Lie derivatives in the modern literature is what it is, we now proceed to show how a Lie operator as defined by Kähler (and by Cartan, except that he did not use this terminology but infinitesimal operator) can be reduced to a partial derivative.

Consider the differential system

$$\frac{\partial x^i}{\partial \lambda} = \alpha_i(x^1, \; ... \; x^n), \tag{1.36}$$

the α_i not depending on λ. One of n independent "constant of the motion" (i.e. line integrals) is then additive to λ. It can then be considered to be λ itself. Denote as y^i $(i = 1, \quad n-1)$ a set of $n-1$ such integrals, independent among themselves and independent of λ, to which we shall refer as y^n. The y^i's $(i = 1, \quad n)$ constitute a new coordinate system and we have

$$x^i = x^i(y^1, \quad , \; y^n). \tag{1.37}$$

In the new coordinate system, the Lie operator reads $X = \beta^i\partial/\partial y^i$. Its action on a scalar function is

$$\beta^i\frac{\partial f}{\partial y^i} = \alpha^l\frac{\partial f}{\partial x^l} = \frac{\partial x^l}{\partial \lambda}\frac{\partial f}{\partial x^l} = \frac{\partial f}{\partial y^n}. \tag{1.38}$$

We rewrite u (given by (1.5)), as

$$u = \frac{1}{p!}a_{i_1...i_p}\frac{\partial x^{i_1}}{\partial y^{i_1}} \quad \frac{\partial x^{i_p}}{\partial y^{i_p}}dy^{k_1}\wedge ... \wedge dy^{k_p}, \tag{1.39}$$

and then

$$\frac{\partial u}{\partial y^n} = \frac{1}{p!}\frac{\partial a_{i_1...i_p}}{\partial y^n}\frac{\partial x^{i_1}}{\partial y^{i_1}} \quad \frac{\partial x^{i_p}}{\partial y^{i_p}}dy^{k_1}\wedge ... \wedge dy^{k_p} + \\ + \frac{1}{(p-1)!}a_{i_1...i_p}\frac{\partial}{\partial y^n}\left(\frac{\partial x^{i_1}}{\partial y^{k_1}}dy^{k_1}\right)\frac{\partial x^{i_2}}{\partial y^{k_2}} \quad \frac{\partial x^{i_p}}{\partial y^{k_p}}dy^{k_2}\wedge ... \wedge dy^{k_p}. \tag{1.40}$$

We now use that

$$\frac{\partial a_{i_1...i_p}}{\partial y^n} = \frac{\partial a_{i_1...i_p}}{\partial x^i}\frac{\partial x^i}{\partial y^n} = \alpha^i \frac{\partial a_{i_1...i_p}}{\partial x^i} \tag{1.41}$$

and that

$$\frac{\partial}{\partial y^n}\left(\frac{\partial x^{i_1}}{\partial y^{k_1}} dy^{k_1}\right) = \frac{\partial}{\partial y^{k_1}}\left(\frac{\partial x^{i_1}}{\partial y^n}\right) dy^{k_1} = d\left(\frac{\partial x^{i_1}}{\partial y^n}\right) = d\alpha^{i_1}. \tag{1.42}$$

Hence

$$Xu = \frac{\partial u}{\partial y^n} = \frac{1}{p!}\alpha^i \frac{\partial a_{i_1...i_p}}{\partial x^i} dx^{i_1...i_p} + \frac{1}{p!} a_{i_1...i_p} d\alpha^i \wedge dx^{i_2} \wedge ... \wedge dx^{i_p}. \tag{1.43}$$

and finally

$$Xu = \frac{1}{p!}\alpha^i \frac{\partial u}{\partial x^i} + d\alpha^i \wedge e_i u, \tag{1.44}$$

2 Angular momentum

The components of the angular momentum operators acting on scalar functions are given by (1.3), and therefore

$$\alpha_k = -x^j dx^i + x^i dx^j, \tag{2.1}$$

and

$$d\alpha_k = -dx^j \wedge dx^i + dx^i \wedge dx^j = 2dx^i \wedge dx^j \equiv 2w_k. \tag{2.2}$$

Hence

$$\chi_k u = x^i \frac{\partial u}{\partial x^j} - x^j \frac{\partial u}{\partial x^i} + \frac{1}{2} w_k \vee u - \frac{1}{2} u \wedge w_k. \tag{2.3}$$

The last two terms constitute the component k of the spin operator. It is worth going back to (1.7) and (1.10), where we have the entangled germs of the orbital and spin operator, if we replace $\chi$with χ_k. It does not make sense to speak of spin as intrinsic angular momentum until u represents a particle, which would not be the case at this point.

Kähler denotes the total angular momentum as $K + 1$, which he defines as

$$(K+1)u = \sum_{i=1}^{3} \chi_i u \vee w_i. \tag{2.4}$$

He then shows by straightforward algebra that

$$-K(K+1) = \chi_1^2 + \chi_2^2 + \chi_3^2. \tag{2.5}$$

He also develops the expression for $(K + 1)$ until it becomes

$$(K+1)u = -\sum_i \frac{\partial u}{\partial x^i} \vee dx^i \vee rdr + \sum_i x^i \frac{\partial u}{\partial x^i} + \frac{3}{2}(u - \eta u) + g\eta u \tag{2.6}$$

and also

$$(K+1)u = -\zeta\partial\zeta u \vee rdr + \sum_i x^i \frac{\partial u}{\partial x^i} + \frac{3}{2}(u - \eta u) + g\eta u, \qquad (2.7)$$

where η is as in previous chapters, where ζ reverses the order of all the differential 1−form factors in u and where $g \equiv dx^i \wedge e_i$. This expression for $(K+1)u$ is used in the next section.

3 Strict harmonic differential forms in $E_3 - \{0\}$

This section is a somewhat abbreviated form of Kähler's treatment of strict harmonic differential forms in $E_3 - \{0\}$, meaning the 3-D Euclidean space punctured at the origin of coordinates.

Kähler starts his argument with considerations on Laurent series of harmonic functions. He states that "... one can prove that every time differentiable harmonic function in $E_3 - \{0\}$ can be written as a series"

$$f = \sum_{h=-\infty}^{\infty} f^{(h)}, \qquad (3.1)$$

where $f^{(h)}$ is a homogeneous polynomial of degree h of homogeneity, for $h \geq 0$, and its the product of polynomial by r^{2h+1} for $h < 0$ (A theorem along similar lines in the modern literature can be found in the book "Harmonic Function Theory" by S. Axler, P. Bourdon and W. Ramey, copyrighted in 2001). From there, Kähler argues that one can expand a strict harmonic differential form u as

$$u = \sum_{-\infty}^{\infty} u^{(h)}, \qquad (3.2)$$

where $u^{(h)}$ is (a) a homogeneous of degree h with respect to the Cartesian coordinates, and (b) also being polynomic for $h \geq 0$, and finally the product of a polynomial by r^{-2h-1} for $h < 0$.

We shall consider 3.2 as an ansatz with $u^{(h)}$'s of type (a). As for (b), we shall deal with this in due time. Be aware of the fact that 3.1 is not contained in 3.2, since the former is for harmonic functions and 3.2 is for strict harmonic differential forms. Obviously, non-trivial strict harmonic functions do not exist.

3.1 Simplification by reduction

Kähler shows that there is isomorphisms between modules M_h and M_{-h-2}, for $-\infty < h < \infty$, the subscript h being the degree of homogeneity h

of the members of the module. One can show that the module M_{-1} is empty.

Let u be strict harmonic of degree h. Then,

$$\partial(r^{-2h-2}dr \vee u) = \partial(r^{-2h-2}dr) \vee u + 2[e^l(r^{-2h-2}dr)] \vee d_l u, \tag{3.3}$$

by virtue of the rule for $\partial(v \vee u)$ and the assumption $\partial u = 0$.

Let us compute the first term on the right hand side

$$\partial(r^{-2h-2}dr) = -(2h+2)r^{-2h-3} + r^{-2h-2}\partial dr. \tag{3.4}$$

For ∂dr, we need a little bit of computations which we address using Cartan's notation (If not familiar with it, see this author's "Differential Geometry for Physicists and Mathematicians"). Clearly $\partial dr = \omega^l \cdot d_l dr$, where

$$\omega^1 = dr, \quad \omega^2 = rd\theta, \quad \omega^3 = r\sin\theta d\phi. \tag{3.5}$$

Using

$$d\omega^i = \omega^j \wedge \omega^i_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{3.6}$$

one readily obtains

$$\omega^2_1 = d\theta \quad \omega^3_1 = \sin\theta d\phi, \quad \omega^3_2 = \cos\theta d\phi. \tag{3.7}$$

Then,

$$d_l(dr) = d_l\omega^1 = -\Gamma^1_{l\,i}\omega^i = -\omega^1_l. \tag{3.8}$$

Hence

$$\begin{aligned}\partial(dr) &= \omega^l \cdot d_l(dr) = \omega^2 \cdot (-\omega^1_2) + \omega^3 \cdot (-\omega^1_3) = \\ &= rd\theta \cdot d\theta + r\sin\theta d\phi \cdot \sin\theta d\phi = \frac{1}{r} + \frac{1}{r} = \frac{2}{r}\end{aligned} \tag{3.9}$$

which will go into the last term of (3.4) and then, therefore, into the first term of (3.3). For the second term of (3.3), we use that

$$e^l(r^{-2h-2}dr) = e^l(r^{-2h-3}rdr) = e^l(r^{-2h-3}x^i dx^i) = r^{-2h-3}x^l \tag{3.10}$$

so that it becomes

$$2r^{-2l-3}x^l\frac{\partial u}{\partial x^l} = 2hr^{-2h-3}u, \tag{3.11}$$

after using the homogeneity of u in $d_l u$.

From (3.3), (3.4), (3.9) and (3.11), we get

$$\partial(r^{-2h-2}dr \vee u) = 0. \tag{3.12}$$

hence r^{-2h-2} is strict harmonic. Its degree of homogeneity is $-h-2$. We write the result just obtained as

$$r^{-2h-2}dr \vee u_h = v_{-h-2}. \tag{3.13}$$

No assumption was made as to whether h is positive, or negative, or zero. We proceed to confirm this.

We multiply (3.13) by $r^{2h+2}dr$ and obtain

$$r^{2h+2}dr \vee v_{-h-2} = u_h \tag{3.14}$$

which is equivalent to referring to $-h-2$ as l, in which case, the exponent $2h+2$ becomes

$$2h+2 = 2(-l-2)+2 = -2l-2 \tag{3.15}$$

and

$$r^{-2l-2}dr \vee v_l = u_{-l-2}, \tag{3.16}$$

in agreement with (3.13).

It follows from all this that it suffices to compute M_h for $h \geq 0$ in order to readily obtain M_{-h-2}. It also suffices to compute the subset of even differential forms in M_h since the even ones will be obtained through Clifford multiplication by the unit differential form of grade n.

3.2 Eigen differential forms of total angular momentum

We seek as expansion of strict harmonic differentials by eigen differential forms of the total angular momentum operator, since the equation $\partial_3 u = 0$ has spherical symmetry.

The elements of M_h are of the form $a+v$ where a and v are respectively differential 0-form and 2-form respectively. The formula for the action of K on $a+v$ is

$$Ku = -(h+1)a + [(h+1)v - 2da \wedge rdr] \tag{3.17}$$

Kähler shows that a necessary and sufficient condition for $a+v$ to be an eigen differential of K with proper value k is that the equations

$$-(h+1)a = ka, \qquad (h+1)v - 2da \wedge rdr = kv \tag{3.18}$$

be satisfied. This can be achieved in any of the two following ways:

$$I : a = 0, \qquad k = h+1 \tag{3.19}$$

$$II : k = -h-1, \qquad (h+1)v - da \wedge rdr = kv. \tag{3.20}$$

The solutions of type I are of the form

$$u = df \vee w, \quad \partial\partial f = 0, \tag{3.21}$$

where the annulment of $\partial\partial f$ follows from $\partial u = 0$.

The solutions of type II can be written as

$$u = a + \frac{1}{h+1} da \wedge rdr, \quad \partial\partial a = 0, \tag{3.22}$$

(for the proof of $\partial\partial a = 0$, see further below) and further as

$$\begin{aligned} u &= a + \frac{1}{h+1}(da \vee rdr - da \cdot rdr) = \\ &= a + \frac{1}{h+1}\left(da \vee rdr - \frac{\partial a}{\partial x^i} dx^i \cdot x_j dx^j\right) = a + \frac{1}{h+1}\partial a \vee rdr - \frac{h}{h+1} a. \end{aligned}$$

Hence we have

$$u = \frac{1}{h+1}(a + \partial a \vee rdr), \qquad \partial\partial a = 0, \tag{3.23}$$

as alternative to (3.22).

We now prove that $\partial\partial a = 0$. We use (3.23) and

$$\partial(u \vee v) = \partial u \vee v + (\eta u \vee \partial v) + 2e^i u \vee d_i v,$$

to obtain

$$\begin{aligned} 0 = \partial u &= \partial((h+1)u) = \partial(a + \partial a \vee rdr) = \\ &= \partial a + \partial\partial a \vee rdr - \partial a \vee \partial(rdr) + 2\frac{\partial a}{\partial x^i} \vee d_i(rdr). \end{aligned} \tag{3.24}$$

But the sum of the first, third and fourth term on the right of (3.24) cancel out, since $d_i(rdr) = dx^i$ and

$$\partial(rdr) = \partial(x_i dx^i) = \sum dx^i \vee dx^i = 3. \tag{3.25}$$

So, (3.24) becomes

$$0 = \partial u = \partial\partial a \vee rdr.$$

Clifford multiplication by $\frac{1}{r}dr$ on the right yields $\partial\partial a = 0$.

Notice that we could have used the fact that $h \geq 0$, but have not done so.

The solutions of type II can still be given another form. First notice that if we multiply the right hand side of (3.22) by $h+1$ we reproduce the submodule. So let us write the u in (3.22) and (3.23) further as

$$u = (h+1)a + da \wedge rdr. \tag{3.26}$$

Now observe the following:

$$-r^{2h+2} dr \vee d(r^{-2h-1}a) = (2h+1)a - rdr \vee da, \tag{3.27}$$

but

$$dr \vee da = dr \wedge da + \sum \frac{x^i}{r}\frac{\partial a}{\partial x^i} = -da \wedge dr + \frac{h}{r}a. \tag{3.28}$$

Hence, from (3.27) and (3.28)

$$-r^{2h+2}dr \vee d(r^{-2h-1}a) = (2h+1)a + da \wedge rdr - ha = (h+1)a + da \wedge rdr = u, \tag{3.29}$$

where we have used (3.26) for the last step..

From this equation follows a very important result. Notice that equation (3.13) states that the Clifford product by $r^{-2h-2}dr$ of a strict harmonic differential form of degree of homogeneity h gives a strict harmonic differential form of degree $-h-2$. In (3.29), we solve for $d(r^{-2h-1}a)$ through multiplication by $-r^{-2h-1}dr$ and, since u is homogeneous of degree h, we conclude that $d(r^{2h-2}a)$ is strict harmonic of degree $-h-2$.

Kähler obtains the same result in a different way. Assume that the a's are harmonic functions. Since $r^{-2h-1}a$ is obtained from a by replacement of x^i by x^i/r^2 followed by division by r, it also is harmonic. Hence, it follows from $\partial\partial(r^{-2h-1}a) = 0$ that $\partial(r^{-2h-1}a)$, which equals $d(r^{-2h-1}a)$, is strict harmonic of degree $-h-2$. This leads Kähler to write the submodule of even differential forms as

$$\frac{1+\eta}{2}M_h = dF_{h+1} \vee w + r^{2h+2}dr \vee dF_{-h-1}, \qquad h \geq 0, \tag{3.30}$$

where F_h is the module, say over the complex field, of homogeneous harmonic polynomials of degree h, and F_{-h-1} is the set of harmonic functions $r^{-2h-1}a$.

The members of the module (3.30), but without the input of F_h for $h \geq 0$ being harmonic polynomials, would simply be written as

$$u_h = da_h \vee w + r^{2h+2}dr \vee d(r^{-2h-1}b_h), \tag{3.31}$$

where a_h and b_h are arbitrary harmonic functions. As we just said after Eq.(3.25), h could be ≥ 0 or < 0. We have not assumed one or the other. As for the odd differential forms, they would be given as

$$u_h = de_h + r^{2h+2}dr \vee d(r^{-2h-1}f_h) \vee v, \tag{3.32}$$

where e_h and f_h are arbitrary harmonic functions.

3.3 Angular factor of strict harmonic differential form solutions of $\partial_3 u = 0$

In chapter 2, we already saw that one can accommodate less symmetry in solutions of equations than on the equations themselves. We now look at a similar situation from another perspective.

The Laplacian operator is isotropic. Its solutions are not so in general. In other words, they are not spherically symmetric because of their dependence on θ and ϕ. So, how does the symmetry of an equation reflect itself in its solutions. The answer for Kähler equations with, for example, spherical symmetry,

$$\partial u = a \vee u, \tag{3.33}$$

means that, when we obtain ∂u for any solution, we not recover the solution left multiplied by some spherical symmetric u.

With this observation, we can understand Kähler's construction of solutions from homogeneous harmonic polynomials, namely

$$F_k = \sum_{m=-l}^{l} C \cdot r^k Y_k^m, \qquad l = |k| \tag{3.34}$$

where C stands for complex coefficients C_l^k. The $d(r^k Y_k^m)$ are strict harmonic differentials, as per the first term on the right of (3.32) or because $0 = \partial\partial(r^k Y_k^m) = \partial[d(r^k Y_k^m)]$. We do not want solutions of $\partial_3 u = 0$, but of the Kähler equation. So, we seek solutions whose Kähler derivative will behave as specified in the previous paragraph.

He then defines the "spherical differentials", S_k^m,

$$S_k^m = r^{1-k} d(r^k Y_k^m). \tag{3.35}$$

These are differential 1-forms of degree of homogeneity zero. Using that $d(r^k Y_k^m)$ is strict harmonic, we readily obtain

$$\partial S_k^m = \frac{1-k}{r} dr \vee S_k^m. \tag{3.36}$$

It is not as easy to obtain the following important result for later use:

$$d_r S_k^m = 0, \tag{3.37}$$

for later use. The argument goes as follows. The d_h operator is covariant, i.e. it transforms tensorially. Use a superscript to denote the coordinate system with which one is computing. Denote $d_h^{(x)}$ and $d_h^{(y)}$ as v_h and v'_h.

Clearly, $v_i' = (\partial x^l/\partial y^i) v_l$. Thus $d_i^{(y)} = (\partial x^l/\partial y^i) d_l^{(x)}$ and with d_r denoting $d_i^{(y)}$ for $y^i = r$, we have

$$d_r u = \frac{\partial x^i}{\partial r} d_i^{(x)} u = \frac{x^i}{r}\frac{\partial u}{\partial x^i} = \frac{h}{r} u. \tag{3.38}$$

Since S_k^m is homogeneous of degree zero, Eq. (3.37) follows.

3.4 Inclusion of the radial factor

We now seek radial factors R that will satisfy the rotational symmetry around all three axes,

$$X_i R = 0. \tag{3.39}$$

We show below that

$$\partial(R \vee S_k^m) = (\partial R + \eta\zeta R \vee \frac{1-k}{r} dr) \vee S_k^m, \tag{3.40}$$

the angular dependence on the right hand side thus being confined to S_k^m.

In order to satisfy (3.39), one requires R to be of the form $R = R_1 + R_2$, where

$$R_1 = \rho_0 + \rho_1 dr_1, \qquad R_2 = (\rho_2 + \rho_3 dr) \vee w, \tag{3.41}$$

with $\rho_\nu = \rho_\nu(r)$. In turn, we can rewrite this as

$$R = R_1 + R_2 \vee w, \qquad e_\theta R_{1,2} = e_\phi R_{1,2} = 0. \tag{3.42}$$

Using (3.28) and (3.42), we get

$$\partial(R_1 \vee S) = \partial R_1 \vee S + \eta R_1 \vee \partial S, \tag{3.43}$$

and recalling that w is a constant differential and commutes with the whole algebra,

$$\begin{aligned}\partial(R_2 \vee w \vee S) = \partial(R_2 \vee S \vee w) &= \partial(R_2 \vee S) \vee w = \\ &= \partial R_2 \vee S \vee w + \eta R_2 \vee \partial S \vee w \\ &= \partial R_2 \vee w \vee S + \eta R_2 \vee w \vee \partial S.\end{aligned} \tag{3.44}$$

Needless to say that indices k and m are understood everywhere. Since $R_1 - R_2 \vee w = \tau R$, we further get, using (3.43) and (3.44)

$$\partial(R \vee S) = \partial R \vee S + \eta\zeta R \vee \partial S, \tag{3.45}$$

which, together with (3.36), implies (3.40).]

3.5 The general solution

The S_k defined by

$$S_k = \sum_{m=-l}^{+l} C \cdot S_k^m \qquad (l = |k|) \tag{3.46}$$

constitutes a C-module. Then, by virtue of (3.34) and (3.35),

$$dF_k = r^{k-1} S_k \tag{3.47}$$

and, then, following (3.30) and that

$$\frac{1-\eta}{2} M_h = \frac{1+\eta}{2} M_h \vee w, \tag{3.48}$$

(equivalently eqs, (3.31) and (3.32)), we finally get

$$M_h = r^h S_{h+1} + r^h dr \vee S_{-h-1} + r^h S_{h+1} \vee w + r^h dr \vee S_{-h-1} \vee w. \tag{3.49}$$

(Notice that the first and last terms on the right are from (3.32) and the other one from (3.31). This is, as we argued, valid for $h \geq 0$ and $h < 0$. Kähler had assumed $h \geq 0$. One then Clifford-multiplies (3.49) by $r^{-2h-2} dr$ and obtains

$$\begin{aligned} M_{-h-2} = {} & r^{-h-2} dr \vee S_{h+1} + r^{-h-2} S_{-h-1} + \\ & + r^{-h-2} dr \vee S_{h+1} \vee w + r^{-h-2} w \vee S_{-h-1}. \end{aligned} \tag{3.50}$$

Of course, the form of the right hand side of (3.49) is identical to the form of the right hand side of (3.50). Hence, once again, (3.49) is valid regardless of whether h is positive, zero or negative.

The expansion of M_h leads to the following expansion over the indices K and m

$$u = \sum_{k,m} R_m^k \vee S_k^m \tag{3.51}$$

$$R_k^m = r^k (a_{km} + b_{km} dr + c_{km} w + f_{km} dr \vee w) \tag{3.52}$$

with constant coefficients a_{km}, b_{km}, c_{km} and f_{km}.

4 The fine structure of the hydrogen atom

It takes no extra effort to let the charge of the nucleus be $Z|e|$. This amounts to neglecting the interaction of the electrons in the atom.

The electromagnetic potential then takes the simple form

$$\omega = -c \Phi dt, \qquad \Phi = \frac{Z|e|}{r}. \tag{4.1}$$

The Kähler equation now is

$$\partial u = \frac{1}{\hbar c}(-E_0 + \frac{Zc^2}{r} icdt) \vee u, \tag{4.2}$$

where E_0 is the mass of the electron. We apply to (4.2) the treatment for stationary solutions reported in the previous chapter with

$$u = p \vee e^{-\frac{iEt}{\hbar c}} \epsilon^- \tag{4.3}$$

where E is the energy of the electron in the external field. α and β as defined in chapter 4 now take the form

$$\alpha = -\frac{E_0}{\hbar c}, \qquad \beta = \frac{1}{\hbar c}\frac{Ze^2}{r} \tag{4.4}$$

and the equation for p becomes

$$\partial p - \frac{1}{\hbar c}(E + \frac{Ze^2}{r}) \vee \eta p + \frac{1}{\hbar c} E_0 p = 0. \tag{4.5}$$

Because of spherical symmetry, we use the ansatz

$$p = R \vee S_k^m, \tag{4.6}$$

where R is spherically symmetric. From (4.5), (4.6) and (3.40), we obtain

$$[\partial R + \eta\zeta R \vee \frac{1-k}{r} + \frac{1}{\hbar c}(E + \frac{Ze^2}{r}) \vee \eta R + \frac{1}{\hbar c} E_0 R] \vee S_k^m = 0. \tag{4.7}$$

Let us refer to the contents of the square brackets as $\mathcal{R}$ and let us solve (4.7) by setting $\mathcal{R} = 0$. On account of the first of equations (3.42), we write this as

$$\mathcal{R}_1 + \mathcal{R}_2 \vee w = 0, \tag{4.8}$$

where $\mathcal{R}_i$ is $\mathcal{R}$ with R replaced with $R_i (i = 1, 2)$. Hence we proceed to solve the equation

$$\partial R_i + \eta R_i \vee \frac{1-k}{r} dr \pm \frac{1}{\hbar c}(E + \frac{Ze^2}{r})\eta R_i + \frac{1}{\hbar c} E_0 R_i = 0, \tag{4.9}$$

where the top and bottom signs correspond to R_1 and R_2 respectively.

Both R_1 and R_2 are sums of 0-form and 1-form that only depend on r and dr. For later comparison with equations in the physics literature, we write:

$$R_i = f(r)dr - g(r), \tag{4.10}$$

where the subscript i for f and g is understood but not made explicit because the sign $\pm$ to be used now makes it unnecessary.

When computing ∂R_i we encounter ∂dr, which we found to be $2/r$ (see eq. (3.9). We take ∂R_i into (4.9) and set equal to zero both the scalar part and the coefficient of dr. We obtain the two systems

$$\frac{df}{dr} + \frac{1+k}{r} f = \frac{1}{\hbar c}[E_0 \pm (E + \frac{Ze^2}{r})]g, \tag{4.11}$$

$$\frac{dg}{dr} + \frac{1-k}{r} g = \frac{1}{\hbar c}[E_0 \mp (E + \frac{Ze^2}{r})]f, \tag{4.12}$$

respectively for the upper and lower signs. For each of the two systems, we have the sign plus in front of k in the first equation and the minus sign in the second equation. This difference arises from the action of η on the 0-form and 1-form parts of each of the R_i's, and similarly for the change from $\pm$ to $\mp$. The two systems corresponds to the two solutions

$$u = [f(r)dr - g(r)] \vee S_k^m \vee T^- \tag{4.13}$$

with f and g solutions of the first system, and

$$u = [f(r)dr - g(r)] \wedge S_k^m \vee w \vee T^-, \tag{4.14}$$

with f and g solutions of the second system and where

$$T^- \equiv e^{-\frac{iEt}{\hbar c}} \epsilon^-. \tag{4.15}$$

Notice that we can obtain the (first) system

$$\frac{df}{dr} + \frac{1+k}{r} f = \frac{1}{\hbar c}[E_0 + (E + \frac{Ze^2}{r})]g, \tag{4.16}$$

$$\frac{dg}{dr} + \frac{1-k}{r} g = \frac{1}{\hbar c}[E_0 + (E + \frac{Ze^2}{r})]f, \tag{4.17}$$

from the second one

$$\frac{df}{dr} + \frac{1+k}{r} f = \frac{1}{\hbar c}[E_0 - (E + \frac{Ze^2}{r})]g, \tag{4.18}$$

$$\frac{dg}{dr} + \frac{1-k}{r} g = \frac{1}{\hbar c}[E_0 + (E + \frac{Ze^2}{r})]f, \tag{4.19}$$

by the exchange $(f, g) \to (g, f)$ and $k \to -k$. We use this fact to reduce the problem of solving two systems to solving just one by further observing that, up to the sign, we can also obtain this replacement $(f, g) \to (g, f)$ in (4.13) through multiplication by dr, i.e.

$$[f(r)dr - g(r)]dr = -[g(r)dr - f(r)]. \tag{4.20}$$

The factor -1 can be ignored since it is absorbed into the coefficient of S_k^m. And the change in sign of k is absorbed because $-\infty < k < \infty$, or $k = |k|, 0, -|k|$. Hence, we remove the subscript in (4.10) and use R to refer to R_2. We thus write the solutions (4.13) and (4.14) as

$$u = R \vee S_k^m \vee T^- \ \text{ and } \ u = R \vee dr \vee w \vee S_k^m \vee T^-. \tag{4.21}$$

The system (4.18)-(4.19) is well known from the treatment of one-electron atoms with the Dirac equation, as per section 151 of volume XXXV of the Handbuch der Physik, or as the treatment in the book "Quantum Mechanics of One – and Two – electron Atoms" by E. E. Salpeter and H. Bethe. See also many other books on quantum mechanics (Notice that, in those references, the role of k is played by $-\varkappa$). We thus need not go further as the obtaining of the fine structure continues in the standard way.