Grassmannian Algebras and the Erlangen Program with emphasis on Projective Geometry

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Abstract.

Grassmann's legacy is certainly constituted by his many revolutionary concepts and by exterior algebra, rightly attributed to him and which sometimes bears his name. But he also put a foot in the door of Cliifford algebra and, to quote É. Cartan, he also created *a very fruitful geometric calculus* —specially for projective geometry— where both points and vectors pertain to the *first or primitive class*[1]. Grassmann did his work[2] during the golden age of synthetic geometry, which also was the stone age of the algebraic foundations of projective geometry. As we shall show, these foundations are subordinate to those of affine geometry, which is the reason why É. Cartan developed his general theory of connections starting not with the Euclidean or projective ones, but with affine connections. The same will be the case here for the corresponding elementary or Klein geometries, which the theory of the different types of connections generalizes[3].

The use of algebra that respects the equivalence of all points in affine geometry —thus the absence of a "zero point"— leads to the concept of canonical affine frame bundle, where the frames are constituted by a point and a vector basis. But bundles of frames made of points or of lines or, in dimension n, of linear varieties of dimension $(n-1)$, may also be used in affine geometry[4]. This leads us to consider the relation of frame bundles to Klein geometries.

The representation up to a proportionality constant of projective transformations as homographies, which constitute the projective group of matrices, almost fits the Erlangen program. But the subgroup that leaves a point

[1]E. Cartan: Nombres complexes. Encyclop. Sc. math. French edition, I5, 1908.

[2]Herman Grassmann, A New Branch of Mathematics: The Ausdenungslehre of 1844, and Other Works. Open Court, Chicago, 1995.

[3]É. Cartan: Sur les variétées a connexion affine et la théorie de la relativité généralisé, Ann.École Norm. **40**, 325-412 (1923).

[4]R. González-Calvet, Treatise of Plane Geometry through Geometric Algebra , 1996.

unchanged —essential in Klein geometries— and the matrix representation of the affine group are typically overlooked. So has been, therefore, the issue of what synthetic projective transformations are directly related to the post-affine entries in the homographies. We exhibit the subgroup of such transformations and show that the proper homologies —i.e. not involving elements at infinity— are directly related to those entries.

We re-interpret from the canonical frame bundle González's version of Möbius-Grassmann-Peano theory, the usefulness of that bundle being enriched in the process. Thus, his special barycentric coordinates now also belong to a theory of moving frames where one includes "frames that do not move". Improper elements, arising from the use of homogeneous coordinates, are not needed if duality is not taken too far, as when one replaces the statement that "parallel lines do not intersect" with the statement that "they intersect at a point at infinity". It is worth noting that the line at infinity is dual to the centroid of a triangle, which is not a special point. So, duality is a very important correspondence, but does not respect the equivalence of all points (unless, of course, we were to create an unnecessary superstructure that mimicked the bundles of frames). Thus González's treatment of Grassmann's system for projective geometry takes it closer to the theory of the moving frame. Of course, there is nothing moving in this case, since nothing needs to do so in the Klein geometries; only their Cartanian generalizations need that the frames "move".

We proceed to briefly summarize Cartan's derivation of the equations of structure of projective connections[5]

Finally, the Kähler calculus[6] can claim to have Grassmann in its ascendancy. We shall illustrate how it blends Clifford algebra with exterior calculus.

[5]É. Cartan, Sur les varietées à connexion projective, Bull. Soc. math **52,** 205-241, 1924.

[6]E. Kähler, Der innere Differentialkalkül, Rendiconti di Matematica e delle sue Applicazioni, **XXI**, 425-523, 1962.

1 Introduction

1.1 Of algebra and geometry

This introduction deals at an elementary level with the three concepts in the title of the paper. Clifford algebra is not mentioned because it has very little to do with the Erlangen program and projective geometry. In algebra, there are not points. Hence, projective geometry cannot be a by-product of Clifford algebra. In contrast, Grassmann's progressive-regressive system has points and vectors together in the first or fundamental class.

In Grassmann's time, there was not a concept of geometry like in the Erlangen program. The latter was born in 1872, almost as late as his death. Of the same decade is the birth of Clifford algebra, which supersedes the algebraic part of Grassmann's system, not its geometric part, with which it is entangled. If not Clifford algebra, what supersedes the geometric part?

Progress on a better algebraic approach to geometry through algebra required an understanding of the essential difference between algebra and geometry. As we shall explain, É. Cartan must have known this very well, as his authorship of the theory of connections indicates. The key concept for progress was "frame bundles", which are principal fiber spaces. More on this is to be found further below.

As J. Dieudonné wrote on Cartan in context of Riemannian spaces ("here" in the quotation that follows"):

> "Finally, it is fitting to mention the most unexpected extension of Klein's ideas in differential geometry... By an extremely original generalization, É. Cartan was able to show here as well that the idea of "operation" still plays a fundamental role; but it is necessary to replace to replace the group with a more complex object, called the "principal fiber space"; one can roughly represent it with a family of isomorphic groups, parametrized by the different points under consideration; the action of each of these groups attaches objects of an "infinitesimal nature" (tangent vectors, tensors, differential forms) at the same point; and it is by "pulling up to the principal fiber" that É. Cartan was able to inaugurate a new era in the study (local and global) of Riemannian spaces and their generalizations.

We have got this quotation from the introduction to the book "The

Method of Equivalence" by R. Gardner, who credits his assistant Adam Falk with the translation from Dieudonné's introduction to *The Erlangen Program* by Felix Klein.

We return to the relation between Grassmann's system and the modern version of the Erlangen program. In his 1908 paper, *Complex Numbers,* Cartan used 17 of its 146 pages to describe Grassmann's work, mostly the latter's progressive-regressive system. As soon as two years later, two of this paper's sections carried "moving trihedron" in their titles. He might instead have used the retrospectively obvious term "moving frames". He used frames and their bundles to introduce algebraic structure in geometries.

Cartan chose affine geometry as the most relevant one for his first generalization of any Klein geometry. He did so in a way consistent with is view that, for each qualified Lie group, there is a geometry which is to it what Euclidean geometry is to the Euclidean group. He did not choose the more general projective geometry, or the more pertinent Lorentz-Minkowski geometry, as he could have done given that he presented general relativity as if it were the motivation of his work on a general theory of connections.

Affine spaces are the most general manifolds that, without improper elements, are globally associated with vector spaces. These are a matter of algebra and have a special element, the zero. Affine spaces are a matter of geometry and do not have a special element, no special point, no "zero". These few statement is all that one needs to understand Cartan's generalization of Klein geometries, and to fill gaps in the study of projective geometry from the perspective of the Erlangen program.

1.2 Of Grassmannian algebras and geometry

Exterior algebra is the name of the algebra for which Grassmann is best known by the general mathematical public. But this recognition is a meager favor to him since it distracts from his many significant mathematical contributions. Speaking specifically of algebras, his formulation of a multitude of new products virtually amounts to his introduction of informal quotient algebras (of equally informal tensor algebras) by binary relations. But his most important algebraic work was his exterior-interior system.

A comment by Dieudonné in his paper *The Tragedy of Grassmann* seems pertinent here: "Grassmann is primarily interested in n-dimensional geometry, and not in algebra...". Here, we are not interested in whether this quoted statement is correct or not, but in that it implicitly states the need to distinguish between these two branches of mathematics. The distinction appears to have been ignored by many modern algebraists.

Grassmann's system certainly has much to do with geometry, as can be inferred from Dieudonné's comment. By far, the biggest name in geometry since the Ausdehnungslehre is É. Cartan. he was able to distinguish very clearly between algebra (under the name of calculus) and geometry in Grassmann's work. In the aforementioned 17 pages, he had this to say about Grassmann's system: "If one applies to geometry the extensive calculus ... of which H. Grassmann's has developed the laws, one obtains a very fruitful *geometric analysis.* (Emphasis added).

Fifty four pages later Cartan devoted five pages to "The systems of complex numbers and the groups of transformations". The title "Complex Numbers" of the paper could have been "Algebras that Generalize the Complex Numbers Systems". And given his view —which is the modern one— on the relation of groups of transformations to the concept of geometry, he might have titled those five papers as "The relation of algebra to geometry."

Of special interest here is that paper's section "The systems of Clifford and of Lipschitz". Cartan viewed their work as purely algebraic, which we mention here to insist on the difference between algebra and geometry. This difference is most easily mixed in Euclidean geometry because the closest link to an algebraic representation of geometry that there was in Grassmann's times was analytic geometry in 3-D Euclidean space. The latter is too often misidentified with 3-D Euclidean vector space, the base space of Clifford algebra for that dimension and signature.

In his monumental geometric work, Cartan used exterior algebra and the dot product of vectors. He did not need more than those products for developing the theory of connections.

1.3 The Erlangen program

In the original Erlangen program, a geometry was conceived as the study of anything that is left invariant under the transformations of a Lie group. Retrospectively, this was too lose a concept. As per Cartan's reformulation and extension of that program, one has to distinguish between elementary geometries, nowadays called Klein geometries, and their generalizations, which go by the name of the theory of connections.

A Klein geometry is a pair of group, G, and subgroup, G_o, and a property involving them that we shall discuss later on. In Cartan's generalization of

Klein geometry, they are retained but only in differential form (Also for later discussion is the presence of G_o as common group of all the fibers, though the elements of any two different fibers cannot be identified).

Both, Klein geometries and their generalizations share the form of a system of equations known as equations of structure. In the case of Klein geometries, they constitute integrability conditions for another differential system (of connection equations). Upon integration, one obtains the group G. This is not the case for the generalization. Suffice to remember what Dieudonné said of the replacement of the group (G) by an action on a frame bundle. That revolution by Cartan contained another conceptual revolution, namely the following. In the original Erlangen program, Riemannian geometry was the geometry of an infinite Lie group. But Cartan pointed out that not only are such groups not part of what defines a geometry, but they actually mask out what is geometric in geometry. Their presence is not denied; it is just a matter of not assigning them a relevance that they lack.

It follows from these considerations on the superseding by Cartan of the original Erlangen program that he also superseded and made clear the relationship between geometry and algebra present in very entangled form in Grassmann's work.

1.4 Of projective geometry and Clifford algebra

In their paper "Projective Geometry with Clifford Algebra", Hestenes and Ziegler make statements such as:

> "... projective geometry has not been fully integrated into modern mathematics. The reason ... is to be found in incompatibilities of method".

In order to fully integrate projective geometry into modern mathematics, one has to show how it fits into the modern concept of geometry. There is nothing of this sort in their paper, which is not surprising in any case; it is difficult, if not impossible, to find in the literature statements about some simple transformations that complement the affine transformations to yield the projective group for the same dimension. In the process, one should go beyond defining the projective G_o as the subgroup of the projective group that leaves a point unchanged. And that is only a beginning. Later on they state:

> “ ... we seek an efficient formulation of projective geometry with a coherent mathematical system which provides equally efficient formulations of the full range of geometric concepts ...”

The authors should also be specific with regards to “efficiency for what”, certainly not for proving theorems as those by Desargues and Pappus. In his *Treatise on Plane Geometry through Geometric Algebra”*, R. González has already shown how to prove those theorems with just a little exterior algebra, which certainly is a Clifford solution where only its exterior contents is used. The proofs then can be done in the back of an envelope if one first reorganizes his material by dealing with the relation of his frames to those of the canonical frame bundle, and thus to the Erlangen program. The algebraic treatment of those theorems then fit in the back of an envelope.

We celebrate the spirit of both of those quotations, but suggest that proponents of the Hestenes-Ziegler approach to projective geometry go back to the drawing board and tell us what their full paraphernalia of concepts is needed for, after R. González has shown what they are not needed for. On the other hand, I agree with those authors that, in my view, their system has greater advantages than the system of Rota and his followers, though we would certainly wish that these made their counter argument, not only to the Hestenes-Ziegler claim, but also to any claims to be found in this paper and that we have just announced.

Here is the specific claim to be rebuffed by those who may still have a better approach: The Erlangen program is first and foremost what brings order to the rich but disorganized body of projective geometry.