WHAT THE KÄHLER CALCULUS CAN DO THAT OTHER CALCULI CANNOT

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Abstract.

The progress of mathematics makes it sometimes fashionable to describe physical theory in more modern forms, not always deeper or more understandable. Of particular interest in this regard is relativistic quantum mechanics. Modern versions of it, like through the use of geometric calculus (mainly the work of Hestenes), may be more appealing than the original version of Dirac theory, but the physical contents remains virtually unchanged.

Enter the Kähler calculus (KC). Underlied by Clifford algebra of differential forms —like tangent Clifford algebra underlies the geometric calculus— it brings about a fresh new view of quantum mechanics. This view arises, almost without effort, from the equation which is in KC what the Dirac equation is in traditional quantum mechanics. One does not need to first hypothesize foundations of quantum mechanics, which makes the Dirac version unintelligible even when one is adept at computing with it. Many foundations come in the wash from the mathematics and very little additional input. Not only Kähler theory reproduces the main Dirac-related results more elegantly, but does so far more profoundly and shows the way to further developments. In this paper, we shall deal with differences between the Dirac and Kähler versions and, to a lesser extent, between Kähler and Hestenes.

We limit ourselves to scalar-valued differential forms. That is all that one needs to supersede the Dirac and geometric calculus versions of relativistic quantum mechanics. Hence, we shall give just a very brief inkling of KC with post-scalar-valued differential forms. Non-scalar-valuedness is needed for a unification of quantum and classical physics since the curvature and Einstein tensors are inadequate representations of what by their very nature are bivector-valued differential 2-forms and vector-valued differential 3-forms respectively.

We shall be specific about gems, both mathematical and (mainly) physical, contained in this calculus. We shall also explain the mathematical philosophy of Kähler on a variety of issues (vector fields, differential forms, Lie

differentiation, unification of derivatives, product of tangent algebras with algebra of integrands). Not surprisingly, his philosophy is the same as É. Cartan's. We shall also illustrate how some of the results that one achieves with KC supersedes the less sophisticated results which one finds deep into very specialized books on group theory, harmonic function theory, complex variable theory, cohomology theory, relativistic quantum mechanics and even particle theory.

1 Introduction

This paper is being written in response to the unawareness of what the Kähler calculus (KC) has to offer, specially for the advancement of the physics paradigms. In section 2, we speak of this offer in connection with quantum physics, and, in section 3 in connection with physics unification, though only in relation to great overlooked ideas of the 20th century that appear to come together under the KC.

It is a calculus where deep results, both in physics (Section 4) and mathematics (Section 5) follow from just a few definitions. It has not received the attention it deserves. It may be due to the language barrier (papers in German), or to a style that is no longer in use, or, in quantum physics, to misinterpretations, as shown in section 6. In section 7, we shall compare the Kähler calculus with the geometric calculus. In section 8, we explain the mathematical philosophy of Kähler, as it can help understand his work. In section 9, we enumerate a few specialized mathematics and physics books some of whose results are matched or superseded by relatively short proofs that do not require previous knowledge of the subject matter of those books.

2 Brief highlight of Kähler's Quantum Mechanics, meant for quantum physicists

This section is a very brief description of Kähler's Quantum Mechanics, which is a virtual concomitant of the KC. A more detailed description is to be found in Sec. 4.

Assume that you knew KC but not physics and that you asked yourself to solve equations of the form

$$\partial u = au, \tag{1}$$

where we do not need to know details other than ∂ is some Dirac-type derivative operator, that a and u are differential forms (respectively input and output) and that their juxtaposition means their Clifford product. Solutions u may be members of the Clifford algebra which are not necessarily members of ideals in this algebra. Assume further that, to start with, you choose a to be of the type $m+eA$ where A is a differential 1-form and where m and e are constants. Seek solutions in ideals $u\epsilon^{\pm}$ defined by idempotents $\epsilon^{\pm}$ related to time translation symmetry, $\epsilon^{\pm} = \frac{1}{2}(1 \mp idt)$. You can absorb dt in $\epsilon^{\pm}$, and, in particular the dt's in ∂u and au in any of the two equations

$$\partial(u\epsilon^{\pm}) = (m + eA)(u\epsilon^{\pm}). \tag{2}$$

The imaginary factor i has been ignored to avoid distractions, but should be there. clean this equation to, mainly, absorb dt and to leave just $\partial(u\epsilon^{\pm})/\partial dt$ on the left. With little effort, you get the Pauli–Dirac equation as a first approximation, and the Foldy-Wouthuysen transformation in the immediately following second approximation.

Assume on the other hand that, independently of those results, you specialize (2.2) to $A = -(edt/r)$ and solve the equation. You get the fine structure of the hydrogen atom. In neither of these two exercises there is a need for Pauli or Dirac matrices. Nowhere appears the need for negative energy solutions, since what makes the positron a positron is its pertaining to the left ideal defined by ϵ^{+}, not negative energies. Spinors and Hilbert spaces are emerging concepts, as they come in the wash of solving equation (1). Operator theory is not necessary for development of quantum mechanics, as these operators come embedded in the computations, each in its own idiosyncratic way. The concept of probability amplitude also would be an emerging concept, rather than one belonging to the foundations of quantum physics. The reason is that, in the basic Eq. (1), there are not even particles. This has to do with the conservation law, which has to do with two densities at the same time. This has to be seen as pertaining to a magnitude with two opposite signs. clearly this must be charged, as confirmed by results as those of which we have spoken above. This deserves more detailed explanation because it touches very explicitly the Copenhagen interpretation of Quantum Mechanics

A conservation law of a scalar-valued magnitude is usually given the form

$$\frac{\partial \rho}{\partial t} + div\ \mathbf{j} = 0. \tag{3}$$

In quantum mechanics, ρ and $\mathbf{j}$ are built from the wave function (non-relativistic) or the spinor solution of a Dirac equation. In the KC, we can always write

$$u = {}^{+}u\, \epsilon^{+} + {}^{-}u\, \epsilon^{-}. \tag{4}$$

The conservation law obtained from "basic KC theory" (see chapter 6 when posted in this web site) takes the form

$$\frac{\partial \rho_1}{\partial t} + div\ \mathbf{j}_1 + \frac{\partial \rho_2}{\partial t} + div\ \mathbf{j}_2 = 0, \tag{5}$$

where becomes ρ_1 and ρ_2 are at each and every point non-negative and non-positive. Of course, if a system is such that ${}^{+}u$ is zero, you only have "field of negative charge". The probability amplitude is what the negative charge amplitude ${}^{-}u$ looks like.

Probability amplitude thus is a derived concept. It cannot be taken as a basic tenet for an interpretation of quantum mechanics, specially since it was adopted in the conceptual fog that accompanied the birth of quantum mechanics.

3 A new paradigm that the Kähler calculus could make possible

There is synergy between the KC and great overlooked ideas in mathematics and physics in the 20th century, as well as new experimental evidence coming from microelectronics and which contradicts the limited and retrospectively inadequate experimental evidence on which the present theoretical paradigm was built. Imagine that one could put together those ideas in a way consistent with a more refined experimental evidence. That may be possible through the Kähler calculus. You be the judge.

During my first experience as a graduate student, the business of negative energy solutions and an infinite sea of such states made me loose interest in working at the cutting age of physics. I even left graduate school. Years later, I did return to a graduate program. Getting a Ph.D. is much easier when you have a passion for some topic in which there is not much competition, you have time and you have already published in refereed journals. That happened to me. I was a free spirit not constrained by what the National Science Foundation funded. It is a long story. But the success that

I have had in disparate non-main stream field of physics was made possible by job opportunities that no longer exist, and by the support in different ways that Drs. Douglas and Marsha Torr provided me over decades. Thanks to that, I saw new opportunities for superseding physics in more than one area. Eventually, all that came together to form a picture of which I shall try here to give a glimpse. Unfortunately, the world of physics nowadays is a far more hostile environment for free spirits, which I have been. It is thus important to make public new profound scenarios like the one I uncovered, largely because of the right choice of mathematical tools.

Physicists have made great strides with the type of quantum physics with which I had problems. I do not regret this progress. After all, it does not hurt to have the enormous experimental evidence that development of the Dirac theory has brought. But one has to reinterpret it from the perspective of the far move relevant and reliable experimental evidence brought about by the microelectronics revolution. Like the Kähler calculus, it contradicts main tenets of the Bohr-Dirac-Feynman philosophy of quantum physics.

Let me now proceed with some of the great ideas announced in the title of this section, because they are precursors of a new vision of physics, whose developments have barely started.

3.1 Julian Schwinger and source theory

Schwinger's source theory may be seen as a proxy for what the Kähler calculus will become when used to address the same issues. But it has received far less attention than it deserves. Source theory is difficult to define. Its major attractiveness is that the results of quantum electrodynamics are reproduced without the irrelevance of divergent quantities and renormalizations". It emphasizes spacetime, but it is not operator field theory. Of course operators will nevertheless play some role; they also do in the KC, but not as fundamentally as in quantum field theory.

Like S-matrix theory, it also has phenomenological emphasis, which we do not view with enthusiasm. But the phenomenology might look less so when approached with the more formal perspective that the Kähler calculus provides. For instance, what we call the dominant energy-approximation, is not even conceived as an approximation in Dirac's equation where mass is an essential ingredient; it is not so in Kähler's theory, where it comes in because of phenomenology, even if very basic one. The day one shall know enough to actually compute the mass of the electron, we shall be justified in considering masses as pertaining to a higher category than phenomenology.

Schwinger points out that

> "... in general, particles must be created in order to study them, since most of them are unstable. In a general sense this is also true of high-energy stable particles, which must be created in that situation by some device, i.e. an accelerator. One can regard all such creation acts as collisions, in which the necessary properties are transferred from other particles to the one of interest... The other particles in the collision appear only to supply these attributes. They are, in an abstract sense, the source of the particle in question... We try to represent this abstraction of realistic processes numerically..."

And further down, he writes:

> "Unstable particles eventually decay and the decay process is a detection device. More generally, any detection device can be regarded as removing or annihilating the particle. Thus the source concept can again be used as an abstraction ... with the source acting negatively, as a sink."

What is the abstraction? Speaking of the creation of a particle with specified properties in a collision, Schwinger has this to say:

> " ...the source concept is the abstraction of all possible dynamical mechanisms whereby the particular particle can be produced."

A shallow immersion in source theory is all that one needs to realize that it is a calculus of integrals. In the KC, these are evaluations (read integrations) of differential forms. It has the flavor of the KC, not of operator field theory. In the second page where equations are given in his first book on source theory, he states:

> "To specify a weak source , we consider its effectiveness sin crating a particle with momentum $\mathbf{p}$, in the small range $(d\mathbf{p})$.An invariant measure of momentum space is
>
> $$d\omega_p = \frac{(d\mathbf{p})}{(2\pi)^3}\frac{1}{2p^0,}, \qquad\qquad p_0 = +\sqrt{\mathbf{p}^2 + m^2}.$$

We now define the source K in terms of the creation and annihilation probability amplitudes

$$\langle 1_p \mid 0_- \rangle^K = \sqrt{d\omega_p} iK(p),$$

$$\langle 0_+ \mid 1_p \rangle^K = \sqrt{d\omega_p} iK(-p),$$

which conveys the idea that the source liberates or absorbs momentum $\mathbf{p}$ in the respective processes."

Of course, this is not Kähler notation but certainly illustrates the role that integrands play in source theory, as in the KC. Also significant is his principle of unity of the source, because it embodies a postulate with the same flavor as the Kähler equation in his general form, i.e. before we introduce a mass term (or any other specific term) in the input a. We shall come back to this. Let us say what Schwinger says in this regards. He considers what he calls a complete situation, namely one where particles are created by sources K_2, propagate in space and time and are detected by K_1. After a small computation yielding

$$\begin{aligned}\langle 0_+ \mid 0_- \rangle^K \;\; &\cong \;\; 1 + O(K^1)^2 + O(K^1)^2 \\ &+ \int d\omega_p \int (dx)(dx')iK_1(x)e^{ip(x-x')}iK_2(x'),\end{aligned}$$

he states:

"We regard $K_1(x)$ and $K_2(x)$ as manifestations of the same physical mechanism, that is, they are the values of one general source in different spacetime regions. Therefore the only possible combination that can occur is the total source

$$K = K_1 + K_2.$$

This is a fundamental postulate, the principle of the unity of the source, which embodies the idea of the uniformity of nature."

This disquisition by Schwinger has the flavor of what the general Kähler equation is. Mass can only enter in applications. The Kähler equation that one should dream of has to be one where mass does not enter, since it must be valid anywhere. But practical calculations should not take place with that dream equation, certainly not if it is not even known For practical calculations, we choose an input, like when we use the dominant energy approximation. We shall hear a lot about this concept in the KC because this is what relativistic quantum mechanics is about (See chapter 4 in the web site of the Alterman event). Schwinger toys with the same idea, but at the level of interactions, rather than at a level of detail evolution of a system.

3.2 Carver Mead and the concepts of electrons and photons

Let us next deal with the conceptual revolution subjacent in the explosion of microelectronics knowledge of the last half a century. Carver Mead is emeritus professor at Caltech, main brain behind this microelectronics revolution. He is the 1999 Lemelson-MIT Prize for Invention and Innovation. But that is only a very small description of his many credits. Please google his name. He has in common with Schwinger that they are against the paradigm's description of the quantum world in terms of point particles and operators.

For Mead, an electron has the property of adapting to its environment, be it a hydrogen atom or a wire. He claims that experiments are regularly performed with neutrons that are one foot across. In his laboratory, he can make electrons that are ten feet long. He makes statements like "The electron ... is the thing that is wiggling, and the wave is the electron". His use of the term wave is not the standard one of classical electrodynamics (He would use the term non-coherent rather than classical). This characterization of electrons as waves is crucial in order not to misunderstand him in what follows.

Asked what should we think of a photon, Mead had this to say: "*John Cramer at the University of Washington was one of the first to describe it as a transaction between two atoms*". Then he was asked: "So that transaction is itself a wave?" Response: "The field that describes that transaction is a wave, that is right".

This vision of microphysics is totally at odds with present Heisenberg-Dirac-Feynman type formulation of quantum physics, and of cutting edge theories based on auxiliary bundles not directly related to the tangent bundle. It is consistent with the view that we have spoused at the end of the previous subsection. There is not such a think as point particles. These are regions of space where the "background field" presents some concentrations of some sort, with some well defined algebraic (member of an ideal) or topologic-geometric invariant. To think of a photon as some kind of particle is stretching things too far. Things looked that way a century ago, when experimental technique was so primitive, as Mead would argue. And who could speak more authoritatively than he did about situations in an electronics laboratory? The photon is the field that is being transacted There will be some region —fuzzy or not, changing or not, it does not matter— where the transaction is taking place, not just a point of impact?

Those auxiliary bundles are directly tied to the point particle approach of modern physics since you cannot do tangent bundle physics, i.e. regular differential geometry, with them. They are a back road to the ideal of having theoretical physics be differential geometry. After all, is that not what Yang-Mills theory seeks to do? The problem is: what is behind those auxiliary bundles to which Yang-Mills theory resorts? Is there a need for them, or it is simply a matter of using them because one does not know better?. One certainly cannot do better if one does not first imagine what things could look like.

KC for quantum physics is in tune with a hypothetical geometrization of classical physics directly related to the tangent bundle. So it provides the connection between differential geometry and quantum physics. Why is that so?

From the equations of structure of a differential manifold, only the torsion is available for the electromagnetic field. In Riemann-plus-torsion geometry, one cannot match a two-index quantity (here electromagnetic field), with one which has three indices (here torsion). But the matching is possible in Finslerian structures. One can match the two differential form indices of the torsion with those of the electromagnetic field. Hence the 4-potential must be viewed as a differential 1-form, not as a 4-vector. This advocates Kähler's quantum mechanics, not Dirac's.

We now start to show the path towards the connection between classical and quantum physics.

3.3 É. Cartan and the concept of differential form in electrodynamics

The point just made about the mathematical language for physics is so important that we reinforce the argument for the KC calculus in quantum physics with an overlooked study that Cartan did of this topic in his second paper on the theory of affine connections. After representing the electromagnetic quantities in terms of differential forms, he argued that Maxwell's equations should not be viewed as relations at each point of the components of the quantities that enter those equations, i.e. the differential forms. They should be viewed as relating integrals, which is equivalent to viewing them as relations of integrands, not of antisymmetric multilinear functions of vectors. Does this not have the flavor of sources (extended objects rather than points) and of the Kähler calculus?

With the quotation that follows, Cartan then begins a discussion electromagnetic energy-momentum, where vector-valuedness becomes of the essence:

> "But Maxwell's equations (8) do not provide all the laws of electromagnetism.
>
> One knows that in Lorentz theory there is an electromagnetic energy-momentum that is represented by a sliding *vector*..."

At that point Cartan starts to use differential forms that are not scalar-valued.

Since classical physics requires the use also of non-scalar-valued differential forms (gravitation theory for sure) and classical physics is what it is because quantum physics is what it is, a comprehensive KC and concomitant physics will have to be one for non-scalar-valued differential forms, the essential objects in differential geometry. Kähler did not go with his calculus beyond simply defining a Dirac type equation for tensor-valued differential forms. It is not our intention to go into any valuedness other than scalar-valuedness in this summer school, except in the very last day. And there are not other applications at this point except first results on unification of the non-gravitational interactions and the algebraic representation of leptons and quarks through ideals defined by primitive idempotents, beyond what Kähler did in this regard. But the right decision as to what type of valuedness is required comes from the interplay of Kähler's quantum physics with the classical geometrization of the electromagnetic interaction. That is where Einstein comes in the picture with his attempt at unification with teleparallelism (TP). Einstein's is not the only way. In fact, this author got into TP not as a postulate but through the study of the Lorentz force and Finsler geometry. We shall emphasize the Einstein route because his idea was fabulous and because, in the process, Cartan gets again in the picture.

3.4 Einstein and Cartan on teleparallelism

Einstein postulated teleparallelism, i.e. equality of vectors at a distance, because there was no equality of vectors at a distance in his general relativity theory. There was not equality in 1915, year of the birth of general relativity, because the concept of comparison of vectors at a distance was born with the Levi-Civita connection in 1917. but Levi-Civita's is a path-dependent comparison and does not, therefore, qualify as equality. The practical implementation of such equality is through the postulate of annulment of the affine curvature. The Riemannian curvature of a manifold with a metric remains in place, but only in a metric role, as before 1915, not in a metric role, as after 1917. It is obvious from the Cartan-Einstein correspondence that Cartan did no understand that. Einstein believed that the right choice of geometry would bring about the geometrization of electrodynamics, unification with gravity and possibly an alternative to the quantum physics in the Bohr mold.

Einstein's failed in his attempt at physical unification with the **postulate of TP**. And he did so because he did have only a very vague idea of how to connect TP with his **thesis of** what he called **logical homogeneity** of differential geometry and theoretical physics, and with his **view of particles as special regions of the field.** His postulate, thesis and view were in the right track, as later developments in differential geometry have shown. He did not listen at all to Cartan when, on December 3^{rd} 1929, the latter rightly told him that certain identity used as an equation in physics —– which happens to be the first Bianchi identities when there is TP— had to be present in his system of equations. Neither did he pay much attention when in letter of February 2, 1930, Cartan told him the relation of the Ricci tensor to the torsion and its derivatives; this is information contained in the second equation of structure. Cartan was giving Einstein advice consistent with the thesis of logical homogeneity in almost pure form, which takes the form: make the equations of structure and Bianchi identities part of you field equations. But Cartan was not aware of the relation between electromagnetic field and torsion. The mathematics did not yet exist for understanding this relation. It was created precisely by Cartan a few years later with his theory "of metric-Finsler connections". We have used the prefix "metric" because Finslerian connections exist on Riemannian metrics; the Finslerian character of a geometry resides in the type of fibration of its bundle. Cartan put one foot in the door for an understanding of Finsler bundles, but it was an ignored differential topologist by the name of Yeaton H. Clifton —highly praised by S.S. Chern— who pushed the door wide open. If one is used to the thinking of É. Cartan i.e. doing geometry in the bundle rather than its sections, the Finslerian generalization of standard differential geometry is rather straight forward.

3.5 Yeaton H. Clifton and the geometrization of electrodynamics

Enter Clifton, a rather eccentric mathematician who did not care to publish. To remove this difficulty, Douglas G. Torr and myself published in his name. We did so with the title of one of three papers of which he was the major contributor by far: "Finslerian Structure: the Cartan-Clifton method of the moving frame".

An explanation seems in order here. A reviewer of the second of those papers objected to our having used the name of Clifton in referring to the method, as if Cartan had been the creator of the method. But this is incorrect. The names Serret, Frenet, Demoulin, Ribaucour, Cotton and Darboux come to my mind as having been associated with moving frames. Cartan brought the method to new heights with his use in formulating modern differential geometry, aside from the fact that he was by far the best known of the mathematician associated with it. But this use was not viewed as rigorous and a horrible herd of hackers phagocytized it and replaced it with the present Babel tower of formulations of differential geometry, whose authors have missed the essential ideas of Cartan, and of the Erlangen program, and of the theory of integrability of exterior systems, etc. Clifton made the theory of affine connections rigorous, without loosing any of the flavor in Cartan's original work. But not only that. He also rigorously defined Finslerian connections, regardless of whether a metric is defined on it, or whether it is Riemannian or properly Finslerian. This is why his name should be attached to the so called Cartan's theory of moving frames.

Clifton did not quite realize that the equations of the autoparallels of

Finsler connections contain the Lorentz force, certainly in addition to the gravitational force. Let us explain. A Finslerian torsion is of the form

$$\mathcal{R} = (R^{\mu}_{\nu\lambda}\omega^{\mu} \wedge \omega^{\lambda} + S^{\mu}_{\nu l}\omega^{\nu} \wedge \omega^{l}_{\nu})\boldsymbol{e}_{\mu}, \tag{6}$$

Greek and Latin indices being $(0, 1, 2, 3)$ and $(1, 2, 3)$ respectively. Let us ignore the S terms. The torsion then looks as if it pertains to the standard (non-Finslerian) connections with torsions. But they is not quite correct since neither ω^0 nor $\boldsymbol{e}_0$ mean in the Finsler bundles what they mean in the standard bundles. And here comes the amazing feature. Regardless of what the connection with torsion is, the $R^{i}_{\nu\lambda}$ do not contribute to the equation of the autoparallels; only the $R^{0}_{\nu\lambda}$ do. Furthermore, this equation takes the form of the equation of motion with Lorentz force with Ω^0 playing the role of F. Of course, the factor q/m does not appear, which is the reason why some claim that it is not possible to geometrize the equation of motion of electrodynamics. Where there is a solution, naysayers see a problem. A factor such as as q/m is compatible in principle with geometry because a particle is a configuration of a field (Here is Einstein again) with a well defined torsion, which does not contribute to its own acceleration. Of course, the art of geometrization has not yet reached the degree of sophistication necessary to deal with this issue. And, what a coincidence, as Kähler showed charge and energy are both conserved quantities under time translation symmetry.

3.6 Of Kaluza-Klein space and Cartan-Clifton on Finsler bundles

Here is an inkling of things to come. Enter again E. Cartan. In 1922, he did a straightforward computation that shows that the theory of connections is a theory of just moving frames, not frames and particles in an equal footing. One can get particles in the equations of structure by making propertime, τ, a fifth dimension, not just a concept applicable to curves. That means that the four-velocity, $\mathbf{u}$, is now outside spacetime, since it is the fifth element of a basis of vector tangent to the time-space-propertime manifold. $d\tau$ thus is a horizontal differential form dual to $\mathbf{u}$. And it further happens that the arena of quantum physics is not the subspace (t, x^i) but (x^i, τ). This is so because, for example, a hydrogen atom is identical to itself in spite of its state of inertial motion (that is, in flat spacetime, with constant velocity).

This is similar to what happens in Finsler geometry where we have

$$d\mathbf{u} = d\boldsymbol{e}_0 = \omega_0^i \boldsymbol{e}_i. \tag{7}$$

There is here not only the information that $d\mathbf{u}$ equals $d\boldsymbol{e}_0$, but also that the $d\boldsymbol{e}_0$ and thus $d\mathbf{u}$ are horizontal invariants. For those who are not very knowledgeable in differential geometry suffice to say that, for the standard spacetime bundle, the horizontal invariants are the ω^μ's. For the Finslerian spacetime bundle, they are the ω_i^μ and the ω_0^i (Seven "translation" differential forms, like seven is the dimension of the time-space-propertime manifold). This opens the door for the geometrization of quantum physics and unification with classical physics.

A project for a whole new paradigm lies in front of us. The main ideas have been supplied by a constellation of geniuses: Kähler, Schwinger, Mead, Einstein, E. Cartan and Clifton. Is there motivation for such a project? An increasing number of physicists think that the present paradigm is virtually exhausted. Others, who spend time in laboratories rather than in offices of ivory towers would side with Carver Mead, who, in year 2000, opened the first chapter of his book "Collective Electrodynamics" with the following statements: "It is my firm belief that the last seven decades of the 20^{th} century will go down in history as the dark ages of theoretical physics" (now eight and a half and counting).

4 Gems of the Kähler calculus for mathematical analysis

Kähler calculus contains important contributions to analysis and to the understanding of the foundations of certain areas of mathematics. The rich ensemble of results that he obtained, both in quantum physics and mathematics. The components of his tensor-valued differential forms have three series of indices , two of them of subscripts. This speaks of the subtleties of his treatment, which, for instance, distinguishes between differential forms and antisymmetric multilinear functions of vectors fields. His differential forms are integrands, i.e. functions of $r-$surfaces. In other words, we we have to distinguish between antisymmetric multilinear functions of vector fields and integrands, i.e. functions of r-surfaces. Acting on the first ones, Kähler's (as well as Cartan's) operator d yields covariant derivatives. It yields exterior derivatives acting upon the second ones. Because of Cartan's lesser use of tensor-calculus-like use of components, this distinction was less explicit in his work.

Lie differentiation of differential forms is another case in point. His treatment shows that it is a matter of knowing your partial derivatives well. it, and it helps one to better understand that it is dangerous (I am not saying it should not be done!) to deal with sums of terms involving differentials and where differentiations take place leaving constants different sets of coordinates. The brief incursion that follows in the next paragraph should suffice to illustrate this.

Consider $\frac{\partial}{\partial\phi}dx^l$, where the x^l is a Cartesian coordinate and ϕ is the azymuthal coordinate (say as in the spherical and cylindrical systems as well as in an infinity of other systems) Let us denote the coordinates of any such system as y^i, and let ϕ be y^n. We have

$$\frac{\partial}{\partial\phi}dx^i = \frac{\partial}{\partial y^n}\frac{\partial x^i}{\partial y^l}dy^l = \frac{\partial}{\partial y^l}\frac{\partial x^i}{\partial y^n}dy^l = d\frac{\partial x^i}{\partial\phi} = d\alpha^i, \tag{8}$$

where we have defined α^i as $\partial x^i/\partial\phi$. Let u be $u_l dx^l$ and compute $\partial u/\partial\phi$ where the u_l are functions of the x's. We have

$$\frac{\partial}{\partial\phi}u = \frac{\partial}{\partial\phi}(u_i dx^i) = \frac{\partial u_i}{\partial\phi}dx^i + u_i\frac{\partial dx^i}{\partial\phi}. \tag{9}$$

But

$$\frac{\partial u_i}{\partial\phi}dx^i = \frac{\partial u_i}{\partial x^l}\frac{\partial x^l}{\partial\phi}dx^i = \alpha^l\frac{\partial u}{\partial x^l}. \tag{10}$$

Hence

$$\frac{\partial u}{\partial\phi} = \alpha^l\frac{\partial u}{\partial x^l} + d\alpha^i u_i. \tag{11}$$

If we had followed the same process with a differential form of grade greater than one, we would have obtained

$$\frac{\partial u}{\partial\phi} = \alpha^l\frac{\partial u}{\partial x^l} + d\alpha^i \wedge e_i u. \tag{12}$$

The right hand side is known as the Lie derivative of u by the Lie operator $\alpha^i\partial/\partial x^i$, equivalently, by $\partial/\partial\phi$. But why even bother about defining a concept of Lie derivative? Kähler showed how to find ϕ for any given combination $\alpha_i(x)\partial/\partial x^i$.

Of course, the above is well known by experts on the theory of differential equations. The problem is that they only know these things which any undergraduate student in physics and mathematics should learn. As we shall see in a later section, this treatment of Lie differentiation has implications even for the foundations of quantum mechanics, as it establishes a difference between the Dirac and Kaehler formulations of phase factors for rotational symmetry. This difference raises the issue of what may be behind the $\hbar$. More on this later on (but just a little bit).

He went on to further develop the last equation, whose last two terms are not covariant. He converted the ringt hand side of the equation into the sum of two covariant terms. Then, if the metric does not depend on ϕ, the new terms thus obtained are for the field u what the orbital and spin angular momenta are for particles.

Another piece of promising mathematical analysis is constituted by his obtaining the strict harmonic differentials in 3-D Euclidean space punctured at the point that is used as the origin of coordinates. All that was done by Kähler. We now proceed to report on a couple of interested mathematical results obtained by this author.

Classical analysis theory teaches us that integration in the xy plane on a closed curve around a singularity is independent of the curve. One can thus choose to compute on a circle centered at the origin. The integration then takes place with respect to ϕ because differential 1-forms, α, reduce to

$$a = jd\phi$$

on such curves. Since $(dxdy)^2 = -1$, we have

$$d\phi = \frac{xdy - ydx}{\rho^2} = \frac{x - ydxdy}{\rho^2}dy = z^{-1}dy,$$

where $z = 1/(x + ydxdy)$. Since $d\phi \cdot d\phi = 1/\rho^2$, we further have

$$j = \rho^2(\alpha \cdot d\phi).$$

Given $\alpha(= fdx + gdy)$, we can write it as wdx, which defines w. We then compute

$$j = \rho^2(wdx) \cdot (z^{-1}dy)$$

and obtain $(wz)^{(2)}$ where the superscript (2) means the differential 2-form component (i.e. coefficient) of wz. Use this result in the intended integral and make the radius go to zero. The theorem of residues results.

But there may be much important implications if we were interested in the calculus of complex variables. We have just seen in the plane that a strict harmonic even differential form in the real plane plays the role of a holomorphic function in the complex plane. Then, in 3-D, the mathematics starts to look really interesting. We not only have the imaginary unit $dxdy$, but also $dydz$ and $dzdx$. And there is the "entanglement" of imaginary units since

$$(dxdy)(dydz) = dxdz.$$

In addition, we have a basis of strict harmonic differentials in 3-D and what amounts to a Laurent expansion in them. Hence there is a very geometric brand of calculus of several complex variables in 3-D. Higher dimension would be still more interesting because, in addition, we would have products of imaginary units that are differential forms of higher even grade. For the moment, $D > 3$ is of limited interest in this new vision because, apparently, strict harmonic differentials in higher dimension have not yet been worked out.

Still more relevant is the use of the traditional Helmholtz theorem in its "natural environment", i.e. not of vector fields but of differential 1-forms. It is an ackward theorem for those fields because gradients, curls and divergences and divergences are involved. Gradients and curls of vector fields are replaced with just exterior derivatives in a corresponding theorem for differential $1-$forms. This in turn can be eadily extended to arbitrary grade in Euclidean spaces of dimension $n > r$. Furthermore, Riemannian $r-$manifolds can always be viewed as r-surfaces in Euclidean spaces of dimension $N > n$ for sufficiently high N. Since one cannot then use in the proof of a Helmholtz theorem that a certain integral vanishes (the one that vanishes at infinity when integrating over the whole Euclidean space), an extra term (actually terms) contributes to the final result. One thus obtains a theorem of a Helmholz type but with a harmonic terms contribution, like in Hodge's theorem. But it is far more sophisticated than Hodge's, for one not only is proving the decomposition (closed, co-closed and harmonic), but actually specifying what they are in terms of integrands, as in Helmholtz theorem. So, we strike at the heart of cohomology theory without using co-homology theory.

What we have just reported illustrates that, with the KC calculus, one finds results that match or improve on theorems in last chapters of highly specialized books. One does so virtually without attempting to do so. It is only a matter of trying to compute with the KC what you would compote with other calculi in your everyday work with mathematics. New doors open themselves in front of you. And one achieves everything that we have reported through the use of only scalar-valued differential forms. Imagine what may happen if we were to use Clifford-valued differential forms. Can any other calculus compete with Kähler's?