Mathematical Analysis
Some examples

Example 1

You are driving a car along a straight road, and the total distance traveled in \( t \geq 0 \) hours is given by \( f(t) = t^2 + 50t \) kilometers (so, after \( t = 1 \) hour you drove \( f(1) = 51 \) km, after \( t = 3 \) hours you drove \( f(3) = 159 \) km, aso).

If you want to know your average speed for the first 3 hours of your trip, you can compute

\[
\text{Average speed} = \frac{\text{change in position}}{\text{change in time}} = \frac{f(3) - f(0)}{3 - 0} = 55 \text{ km}.
\]

If you want to know your instantaneous speed at \( t = 3 \) hours (the speed you were driving at \( t = 3 \) hours after starting the trip), you can proceed as follows.

First you determine the average speed for the time interval \([3, 3 + \Delta t]\), that is

\[
\text{Average speed on } [3, 3 + \Delta t] = \frac{f(3 + \Delta t) - f(3)}{3 + \Delta t - 3} = \frac{(3 + \Delta t)^2 + 50(3 + \Delta t) - (3^2 + 50 \cdot 3)}{\Delta t} = 56 + \Delta t.
\]
Some examples

Example 1

You are driving a car along a straight road, and the total distance traveled in \( t \geq 0 \) hours is given by \( f(t) = t^2 + 50t \) kilometers (so, after \( t = 1 \) hour you drove \( f(1) = 51 \) km, after \( t = 3 \) hours you drove \( f(3) = 159 \) km, aso).

If you want to know your average speed for the first 3 hours of your trip, you can compute

\[
\text{Average speed} = \frac{\text{change in position}}{\text{change in time}} = \frac{f(3) - f(0)}{3 - 0} = 55 \text{ km}.
\]

If you want to know your instantaneous speed at \( t = 3 \) hours (the speed you were driving at \( t = 3 \) hours after starting the trip), you can proceed as follows.

First you determine the average speed for the time interval \([3, 3 + \Delta t]\), that is

\[
\text{Average speed on } [3, 3 + \Delta t] = \frac{f(3 + \Delta t) - f(3)}{3 + \Delta t - 3} = \frac{(3 + \Delta t)^2 + 50(3 + \Delta t) - (3^2 + 50 \cdot 3)}{\Delta t} = 56 + \Delta t.
\]
Some examples

Example 1

You are driving a car along a straight road, and the total distance traveled in \( t \geq 0 \) hours is given by \( f(t) = t^2 + 50t \) kilometers (so, after \( t = 1 \) hour you drove \( f(1) = 51 \) km, after \( t = 3 \) hours you drove \( f(3) = 159 \) km, aso).

If you want to know your average speed for the first 3 hours of your trip, you can compute

\[
\text{Average speed} = \frac{\text{change in position}}{\text{change in time}} = \frac{f(3) - f(0)}{3 - 0} = 55 \text{ km}.
\]

If you want to know your instantaneous speed at \( t = 3 \) hours (the speed you were driving at \( t = 3 \) hours after starting the trip), you can proceed as follows.

First you determine the average speed for the time interval \([3, 3 + \Delta t]\), that is

\[
\text{Average speed on } [3, 3 + \Delta t] = \frac{f(3 + \Delta t) - f(3)}{3 + \Delta t - 3} = \frac{(3 + \Delta t)^2 + 50(3 + \Delta t) - (3^2 + 50 \cdot 3)}{\Delta t} = 56 + \Delta t.
\]
Some examples

Example 1

You are driving a car along a straight road, and the total distance traveled in \( t \geq 0 \) hours is given by \( f(t) = t^2 + 50t \) kilometers (so, after \( t = 1 \) hour you drove \( f(1) = 51 \) km, after \( t = 3 \) hours you drove \( f(3) = 159 \) km, aso).

If you want to know your average speed for the first 3 hours of your trip, you can compute

\[
\text{Average speed} = \frac{\text{change in position}}{\text{change in time}} = \frac{f(3) - f(0)}{3 - 0} = 55 \text{ km}.
\]

If you want to know your instantaneous speed at \( t = 3 \) hours (the speed you were driving at \( t = 3 \) hours after starting the trip), you can proceed as follows.

First you determine the average speed for the time interval \([3, 3 + \Delta t]\), that is

\[
\text{Average speed on } [3, 3 + \Delta t] = \frac{f(3 + \Delta t) - f(3)}{3 + \Delta t - 3}
= \frac{(3 + \Delta t)^2 + 50(3 + \Delta t) - (3^2 + 50 \cdot 3)}{\Delta t}
= 56 + \Delta t.
\]
Next, you notice that if $\Delta t$ is very small, then the average speed on $[3, 3 + \Delta t]$ is approximately equal to the instantaneous speed at $t = 3$.

To make this approximation exact, you let $\Delta t \to 0$, so you obtain that the instantaneous speed at $t = 3$ is

$$\text{Instantaneous speed at time } 3 = \lim_{\Delta t \to 0} \frac{f(3 + \Delta t) - f(3)}{3 + \Delta t - 3} = \lim_{\Delta t \to 0} (56 + \Delta t) = 56 \text{ km/h}.$$
Next, you notice that if $\Delta t$ is very small, then the average speed on $[3, 3 + \Delta t]$ is approximately equal to the instantaneous speed at $t = 3$.

To make this approximations exact, you let $\Delta t \rightarrow 0$, so you obtain that the instantaneous speed at $t = 3$ is

\[
\text{Instantaneous speed at time 3} = \lim_{\Delta t \to 0} \frac{f(3 + \Delta t) - f(3)}{3 + \Delta t - 3} = \lim_{\Delta t \to 0} (56 + \Delta t) = 56 \text{ km/h}.
\]
Example 2

Suppose that you are given a function $f$ with the graph given below, and you want to find the slope of the tangent line at the point $P(a, b)$.

First, you can determine the slope of the line passing through the point $P(a, b)$ and another point $Q(x, y)$ on the graph of $f$, like this:

\[
\text{slope}_{PQ} = \tan \alpha = \frac{\text{change in } y}{\text{change in } x} = \frac{y - b}{x - a} = \frac{f(x) - f(a)}{x - a}.
\]

**Figure**: Finding the slope of the tangent line to the graph of a the function $y = f(x)$. 

M. N. Pascu (Transilvania Univ)
Example 2

Suppose that you are given a function $f$ with the graph given below, and you want to find the slope of the tangent line at the point $P(a, b)$.

First, you can determine the slope of the line passing through the point $P(a, b)$ and another point $Q(x, y)$ on the graph of $f$, like this:

$$\text{slope}_{PQ} = \tan \alpha = \frac{\text{change in } y}{\text{change in } x} = \frac{y - b}{x - a} = \frac{f(x) - f(a)}{x - a}.$$
To find the slope of the tangent line, you can choose the point $Q(x, y)$ closer and closer to the point $P(a, b)$, that is you find the limit of the previous slope as $x \to a$, like this

\[
\text{slope at } a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.
\]  

(1)

If this limit exists, its value (denoted $f'(a)$) gives the value of the tangent line at $x = a$. Using the point-slope formula for the equation of a line you can now determine the equation of the tangent line to the graph of $f$ at the point $P(a, b)$

\[
y - b = f'(a)(x - a).
\]  

(2)
Example 2

To find the slope of the tangent line, you can choose the point $Q(x, y)$ closer and closer to the point $P(a, b)$, that is you find the limit of the previous slope as $x \to a$, like this

$$\text{slope at } a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}. \quad (1)$$

If this limit exists, its value (denoted $f'(a)$) gives the value of the tangent line at $x = a$.

Using the point-slope formula for the equation of a line you can now determine the equation of the tangent line to the graph of $f$ at the point $P(a, b)$

$$y - b = f'(a)(x - a). \quad (2)$$
Example 2

To find the slope of the tangent line, you can choose the point \( Q (x, y) \) closer and closer to the point \( P (a, b) \), that is you find the limit of the previous slope as \( x \to a \), like this

\[
\text{slope at } a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.
\]  

(1)

If this limit exists, its value (denoted \( f'(a) \)) gives the value of the tangent line at \( x = a \). Using the point-slope formula for the equation of a line you can now determine the equation of the tangent line to the graph of \( f \) at the point \( P (a, b) \)

\[
y - b = f'(a) (x - a).
\]  

(2)
The derivative of a real-valued function of a real variable

Definition 3

Let \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) and \( x_0 \in A \). We say that \( f \) is differentiable at \( x = x_0 \) if the limit

\[
    f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},
\]

exists and it is finite. If the above limit exists, its value is called the derivative of \( f \) at \( x_0 \). If \( f \) is differentiable at any point \( x_0 \in A \), we say that the function \( f \) is differentiable on \( A \) and call the function \( f' : A \subset \mathbb{R} \rightarrow \mathbb{R} \) the derivative of \( f \).

Example 4

To show that the function \( f(x) = \sqrt{x} \) is differentiable at an arbitrary point \( x_0 > 0 \), we compute

\[
    \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} = \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}},
\]

which exists and it is finite for any \( x_0 > 0 \). This shows that \( f(x) = \sqrt{x} \) is differentiable on \((0, \infty)\) and its derivative is \( f'(x) = \frac{1}{2\sqrt{x}} \), \( x > 0 \).
Definition 3

Let $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$ and $x_0 \in A$. We say that $f$ is **differentiable** at $x = x_0$ if the limit

$$
f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0},
$$

exists and it is finite. If the above limit exists, its value is called the **derivative** of $f$ at $x_0$.

If $f$ is differentiable at any point $x_0 \in A$, we say that the function $f$ is **differentiable on** $A$ and call the function $f' : A \subset \mathbb{R} \rightarrow \mathbb{R}$ the **derivative of $f$**.

Example 4

To show that the function $f(x) = \sqrt{x}$ is differentiable at an arbitrary point $x_0 > 0$, we compute

$$
\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \rightarrow x_0} \frac{x - x_0}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} = \lim_{x \rightarrow x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}},
$$

which exists and it is finite for any $x_0 > 0$. This shows that $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$ and its derivative is $f'(x) = \frac{1}{2\sqrt{x}}, x > 0$. 
The derivative of a real-valued function of a real variable

Definition 3

Let \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) and \( x_0 \in A \). We say that \( f \) is differentiable at \( x = x_0 \) if the limit

\[
 f' (x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},
\]

exists and it is finite. If the above limit exists, its value is called the derivative of \( f \) at \( x_0 \). If \( f \) is differentiable at any point \( x_0 \in A \), we say that the function \( f \) is differentiable on \( A \) and call the function \( f' : A \subset \mathbb{R} \rightarrow \mathbb{R} \) the derivative of \( f \).

Example 4

To show that the function \( f(x) = \sqrt{x} \) is differentiable at an arbitrary point \( x_0 > 0 \), we compute

\[
 \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} = \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}},
\]

which exists and it is finite for any \( x_0 > 0 \). This shows that \( f(x) = \sqrt{x} \) is differentiable on \((0, \infty)\) and its derivative is \( f' (x) = \frac{1}{2\sqrt{x}}, \ x > 0 \).
The derivative of a real-valued function of a real variable

Definition 3

Let $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$ and $x_0 \in A$. We say that $f$ is differentiable at $x = x_0$ if the limit

$$f' (x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

exists and it is finite. If the above limit exists, its value is called the derivative of $f$ at $x_0$. If $f$ is differentiable at any point $x_0 \in A$, we say that the function $f$ is differentiable on $A$ and call the function $f' : A \subset \mathbb{R} \rightarrow \mathbb{R}$ the derivative of $f$.

Example 4

To show that the function $f(x) = \sqrt{x}$ is differentiable at an arbitrary point $x_0 > 0$, we compute

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} = \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{\sqrt{x_0}},$$

which exists and it is finite for any $x_0 > 0$. This shows that $f(x) = \sqrt{x}$ is differentiable on $(0, \infty)$ and its derivative is $f'(x) = \frac{1}{2\sqrt{x}}$, $x > 0$. 
Definition 3

Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and $x_0 \in A$. We say that $f$ is differentiable at $x = x_0$ if the limit

$$f' (x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

(3)

exists and it is finite. If the above limit exists, its value is called the derivative of $f$ at $x_0$. If $f$ is differentiable at any point $x_0 \in A$, we say that the function $f$ is differentiable on $A$ and call the function $f' : A \subset \mathbb{R} \to \mathbb{R}$ the derivative of $f$.

Example 4

To show that the function $f (x) = \sqrt{x}$ is differentiable at an arbitrary point $x_0 > 0$, we compute

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{(x - x_0)(\sqrt{x} + \sqrt{x_0})} = \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}},$$

which exists and it is finite for any $x_0 > 0$. This shows that $f (x) = \sqrt{x}$ is differentiable on $(0, \infty)$ and its derivative is $f' (x) = \frac{1}{2\sqrt{x}}$, $x > 0$. 
Example 5

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n \ (n \in \mathbb{N})$. For any $x_0 \in \mathbb{R}$, the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{(x - x_0) \left( x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \ldots + xx_0^{n-2} + x_0^{n-1} \right)}{x - x_0}$$

$$= \lim_{x \to x_0} \left( x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \ldots + xx_0^{n-2} + x_0^{n-1} \right)$$

$$= x_0^{n-1} + x_0^{n-2}x_0 + x_0^{n-3}x_0^2 + \ldots + x_0x_0^{n-2} + x_0^{n-1}$$

$$= nx_0^{n-1},$$

exists and it is finite, and therefore the function $f(x) = x^n$ is differentiable at $x_0$ and $f'(x_0) = nx_0^{n-1}$.

Since $x_0 \in \mathbb{R}$ was arbitrarily chosen, it follows that the function $f(x) = x^n$ is differentiable on $\mathbb{R}$ and its derivative is $f'(x) = nx^{n-1}$.

The appendix at the end of the chapter contains a list of most common differentiable functions and their derivatives.
Example 5

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n (n \in \mathbb{N})$. For any $x_0 \in \mathbb{R}$, the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0) \left( x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \ldots + x_0x^{n-2} + x_0^{n-1} \right)}{x - x_0}$$

$$= \lim_{x \to x_0} \left( x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \ldots + x_0x^{n-2} + x_0^{n-1} \right)$$

$$= x_0^{n-1} + x_0^{n-2}x_0 + x_0^{n-3}x_0^2 + \ldots + x_0x_0^{n-2} + x_0^{n-1}$$

$$= nx_0^{n-1},$$

exists and it is finite, and therefore the function $f(x) = x^n$ is differentiable at $x_0$ and $f'(x_0) = nx_0^{n-1}$.

Since $x_0 \in \mathbb{R}$ was arbitrarily chosen, it follows that the function $f(x) = x^n$ is differentiable on $\mathbb{R}$ and its derivative is $f'(x) = nx^{n-1}$.

The appendix at the end of the chapter contains a list of most common differentiable functions and their derivatives.
Example 5

Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$ ($n \in \mathbb{N}$). For any $x_0 \in \mathbb{R}$, the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{(x - x_0) \left(x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \ldots + xx_0^{n-2} + x_0^{n-1}\right)}{x - x_0}$$

$$= \lim_{x \to x_0} \left(x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + \ldots + xx_0^{n-2} + x_0^{n-1}\right)$$

$$= x_0^{n-1} + x_0^{n-2}x_0 + x_0^{n-3}x_0^2 + \ldots + x_0x_0^{n-2} + x_0^{n-1}$$

$$= nx_0^{n-1},$$

exists and it is finite, and therefore the function $f(x) = x^n$ is differentiable at $x_0$ and $f'(x_0) = nx_0^{n-1}$.

Since $x_0 \in \mathbb{R}$ was arbitrarily chosen, it follows that the function $f(x) = x^n$ is differentiable on $\mathbb{R}$ and its derivative is $f'(x) = nx^{n-1}$.

The appendix at the end of the chapter contains a list of most common differentiable functions and their derivatives.
Remark 6 (Tangent line approximation)

If $f$ is differentiable at $x_0$, from the definition it follows that for values of $x \approx x_0$ close to $x_0$ we have

$$\frac{f(x) - f(x_0)}{x - x_0} \approx f'(x_0).$$

This shows that the function $f$ can be approximated near $x_0$ as follows:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0), \quad \text{for } x \approx x_0. \quad (4)$$

Geometrically this shows that for $x$ close to $x_0$, the values of $f(x)$ can be approximated by the values of the tangent line to the graph of $y = f(x)$ at $x = x_0$ (see Example 2).

Figure: Near $x_0$, the tangent line (red) is a good approximation of the value of the function (blue).
Remark 6 (Tangent line approximation)

If \( f \) is differentiable at \( x_0 \), from the definition it follows that for values of \( x \approx x_0 \) close to \( x_0 \) we have \( \frac{f(x) - f(x_0)}{x-x_0} \approx f'(x_0) \).

This shows that the function \( f \) can be approximated near \( x_0 \) as follows:

\[
f(x) \approx f(x_0) + f'(x_0)(x - x_0), \quad \text{for } x \approx x_0.
\]

(4)

Geometrically this shows that for \( x \) close to \( x_0 \), the values of \( f(x) \) can be approximated by the values of the tangent line to the graph of \( y = f(x) \) at \( x = x_0 \) (see Example 2).

Figure: Near \( x_0 \), the tangent line (red) is a good approximation of the value of the function (blue).
Remark 6 (Tangent line approximation)

If $f$ is differentiable at $x_0$, from the definition it follows that for values of $x \approx x_0$ close to $x_0$ we have 
$$
\frac{f(x) - f(x_0)}{x - x_0} \approx f'(x_0).
$$
This shows that the function $f$ can be approximated near $x_0$ as follows:

$$
f(x) \approx f(x_0) + f'(x_0)(x - x_0), \quad \text{for } x \approx x_0. \tag{4}
$$

Geometrically this shows that for $x$ close to $x_0$, the values of $f(x)$ can be approximated by the values of the tangent line to the graph of $y = f(x)$ at $x = x_0$ (see Example 2).

Figure: Near $x_0$, the tangent line (red) is a good approximation of the value of the function (blue).
We can use the tangent line approximation (formula (4)) to approximate the value of $\sqrt{2}$ as follows.

Consider the function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$, for which $f'(x) = \frac{1}{2\sqrt{x}}, x > 0$.

In this case the tangent line approximation (formula (4)) becomes

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}} (x - x_0),$$

which for $x = 2$ and $x_0 = 1$ gives

$$\sqrt{2} \approx \sqrt{1} + \frac{1}{2\sqrt{1}} (2 - 1),$$

or equivalent $\sqrt{2} \approx 1.5$, which is a reasonably good approximation (recall that $\sqrt{2} = 1.414\ldots$).

To obtain a better approximation of $\sqrt{2}$, we must choose a point $x_0$ closer to 2, for example $x_0 = 1.44$. We obtain in this case

$$\sqrt{2} = \sqrt{1.44} + \frac{1}{2\sqrt{1.44}} (2 - 1.44),$$

and therefore $\sqrt{2} \approx 1.2 + \frac{0.56}{2 \cdot 1.2} = 1.2 + \frac{7}{3} = \frac{43}{3} = 1.433\ldots$
Example 7

We can use the tangent line approximation (formula (4)) to approximate the value of $\sqrt{2}$ as follows.

Consider the function $f : [0, \infty) \rightarrow \mathbb{R}$ given by $f(x) = \sqrt{x}$, for which $f'(x) = \frac{1}{2\sqrt{x}}$, $x > 0$.

In this case the tangent line approximation (formula (4)) becomes

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}} (x - x_0),$$

which for $x = 2$ and $x_0 = 1$ gives

$$\sqrt{2} \approx \sqrt{1} + \frac{1}{2\sqrt{1}} (2 - 1),$$

or equivalent $\sqrt{2} \approx 1,5$, which is a reasonably good approximation (recall that $\sqrt{2} = 1,414 \ldots$). To obtain a better approximation of $\sqrt{2}$, we must choose a point $x_0$ closer to 2, for example $x_0 = 1,44$. We obtain in this case

$$\sqrt{2} = \sqrt{1,44} + \frac{1}{2\sqrt{1,44}} (2 - 1,44),$$

and therefore $\sqrt{2} \approx 1,2 + \frac{0,56}{2\cdot1,2} = 1,2 + \frac{7}{3} = \frac{4,3}{3} = 1,433 \ldots$
Example 7

We can use the tangent line approximation (formula (4)) to approximate the value of $\sqrt{2}$ as follows.

Consider the function $f : [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$, for which $f'(x) = \frac{1}{2\sqrt{x}}$, $x > 0$. In this case the tangent line approximation (formula (4)) becomes

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}} (x - x_0),$$

which for $x = 2$ and $x_0 = 1$ gives

$$\sqrt{2} \approx \sqrt{1} + \frac{1}{2\sqrt{1}} (2 - 1),$$

or equivalent $\sqrt{2} \approx 1, 5$, which is a reasonably good approximation (recall that $\sqrt{2} = 1, 414 \ldots$).

To obtain a better approximation of $\sqrt{2}$, we must choose a point $x_0$ closer to $2$, for example $x_0 = 1, 44$. We obtain in this case

$$\sqrt{2} = \sqrt{1, 44} + \frac{1}{2\sqrt{1, 44}} (2 - 1, 44),$$

and therefore $\sqrt{2} \approx 1, 2 + \frac{0, 56}{2 \cdot 1, 2} = 1, 2 + \frac{7}{3} = \frac{4, 3}{3} = 1, 433 \ldots$
Example 7

We can use the tangent line approximation (formula (4)) to approximate the value of $\sqrt{2}$ as follows.
Consider the function $f: [0, \infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$, for which $f'(x) = \frac{1}{2\sqrt{x}}$, $x > 0$.
In this case the tangent line approximation (formula (4)) becomes

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}} (x - x_0),$$

which for $x = 2$ and $x_0 = 1$ gives

$$\sqrt{2} \approx \sqrt{1} + \frac{1}{2\sqrt{1}} (2 - 1),$$

or equivalent $\sqrt{2} \approx 1, 5$, which is a reasonably good approximation (recall that $\sqrt{2} = 1, 414 \ldots$).
To obtain a better approximation of $\sqrt{2}$, we must choose a point $x_0$ closer to 2, for example $x_0 = 1, 44$. We obtain in this case

$$\sqrt{2} = \sqrt{1, 44} + \frac{1}{2\sqrt{1, 44}} (2 - 1, 44),$$

and therefore $\sqrt{2} \approx 1, 2 + \frac{0, 56}{2 \cdot 1, 44} = 1, 2 + \frac{7}{3} = \frac{4, 3}{3} = 1, 433 \ldots$
Example 7

We can use the tangent line approximation (formula (4)) to approximate the value of $\sqrt{2}$ as follows.

Consider the function $f : [0, \infty) \rightarrow \mathbb{R}$ given by $f(x) = \sqrt{x}$, for which $f'(x) = \frac{1}{2\sqrt{x}}$, $x > 0$.

In this case the tangent line approximation (formula (4)) becomes

$$\sqrt{x} \approx \sqrt{x_0} + \frac{1}{2\sqrt{x_0}} (x - x_0),$$

which for $x = 2$ and $x_0 = 1$ gives

$$\sqrt{2} \approx \sqrt{1} + \frac{1}{2\sqrt{1}} (2 - 1),$$

or equivalent $\sqrt{2} \approx 1.5$, which is a reasonably good approximation (recall that $\sqrt{2} = 1.414\ldots$).

To obtain a better approximation of $\sqrt{2}$, we must choose a point $x_0$ closer to 2, for example $x_0 = 1.44$. We obtain in this case

$$\sqrt{2} = \sqrt{1.44} + \frac{1}{2\sqrt{1.44}} (2 - 1.44),$$

and therefore $\sqrt{2} \approx 1.2 + \frac{0.56}{2.12} = 1.2 + \frac{7}{3} = \frac{4.3}{3} = 1.433\ldots$
Proposition 8 (Properties of differentiable functions)

Let \( f, g : A \subset \mathbb{R} \to \mathbb{R} \) and let \( x_0 \in A \) be a limit point of \( A \). If \( f \) and \( g \) are differentiable at \( x = x_0 \), then:

i) \( f \pm g \) is also differentiable at \( x = x_0 \) and we have

\[
(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)
\]

ii) \( f \cdot g \) is also differentiable at \( x = x_0 \) and we have

\[
(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)
\]

iii) if \( g \neq 0 \) and \( g'(x_0) \neq 0 \) then \( \frac{f}{g} \) is also differentiable at \( x = x_0 \) and we have

\[
\left( \frac{f}{g} \right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}
\]

iv) if \( g(A) \subset B \) and \( h : B \subset \mathbb{R} \to \mathbb{R} \) is differentiable at \( x = g(x_0) \), then the composition \( h \circ g : A \subset \mathbb{R} \to \mathbb{R} \) is differentiable at \( x = x_0 \) and we have

\[
(h \circ g)'(x_0) = h'(g(x_0)) \cdot g'(x_0)
\]
Proposition 8 (Properties of differentiable functions)

Let \( f, g : A \subset \mathbb{R} \rightarrow \mathbb{R} \) and let \( x_0 \in A \) be a limit point of \( A \). If \( f \) and \( g \) are differentiable at \( x = x_0 \), then:

i) \( f \pm g \) is also differentiable at \( x = x_0 \) and we have

\[
(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)
\]

ii) \( f \cdot g \) is also differentiable at \( x = x_0 \) and we have

\[
(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)
\]

iii) if \( g \neq 0 \) and \( g'(x_0) \neq 0 \) then \( \frac{f}{g} \) is also differentiable at \( x = x_0 \) and we have

\[
\left( \frac{f}{g} \right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}
\]

iv) if \( g(A) \subset B \) and \( h : B \subset \mathbb{R} \rightarrow \mathbb{R} \) is differentiable at \( x = g(x_0) \), then the composition \( h \circ g : A \subset \mathbb{R} \rightarrow \mathbb{R} \) is differentiable at \( x = x_0 \) and we have

\[
(h \circ g)'(x_0) = h'(g(x_0)) \cdot g'(x_0)
\]
Proposition 8 (Properties of differentiable functions)

Let \( f, g : A \subset \mathbb{R} \to \mathbb{R} \) and let \( x_0 \in A \) be a limit point of \( A \). If \( f \) and \( g \) are differentiable at \( x = x_0 \), then:

i) \( f \pm g \) is also differentiable at \( x = x_0 \) and we have

\[
(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)
\]

ii) \( f \cdot g \) is also differentiable at \( x = x_0 \) and we have

\[
(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)
\]

iii) if \( g \neq 0 \) and \( g'(x_0) \neq 0 \) then \( \frac{f}{g} \) is also differentiable at \( x = x_0 \) and we have

\[
\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}
\]

iv) if \( g(A) \subset B \) and \( h : B \subset \mathbb{R} \to \mathbb{R} \) is differentiable at \( x = g(x_0) \), then the composition \( h \circ g : A \subset \mathbb{R} \to \mathbb{R} \) is differentiable at \( x = x_0 \) and we have

\[
(h \circ g)'(x_0) = h'(g(x_0)) \cdot g'(x_0)
\]
Proposition 8 (Properties of differentiable functions)

Let $f, g : A \subset \mathbb{R} \to \mathbb{R}$ and let $x_0 \in A$ be a limit point of $A$. If $f$ and $g$ are differentiable at $x = x_0$, then:

i) $f \pm g$ is also differentiable at $x = x_0$ and we have

$$(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$$

ii) $f \cdot g$ is also differentiable at $x = x_0$ and we have

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0)$$

iii) if $g \neq 0$ and $g'(x_0) \neq 0$ then $\frac{f}{g}$ is also differentiable at $x = x_0$ and we have

$$\left( \frac{f}{g} \right)'(x_0) = \frac{f'(x_0) g(x_0) - f(x_0) g'(x_0)}{g^2(x_0)}$$

iv) if $g(A) \subset B$ and $h : B \subset \mathbb{R} \to \mathbb{R}$ is differentiable at $x = g(x_0)$, then the composition $h \circ g : A \subset \mathbb{R} \to \mathbb{R}$ is differentiable at $x = x_0$ and we have

$$(h \circ g)'(x_0) = h'(g(x_0)) \cdot g'(x_0)$$
Example 9

Using the above properties (the quotient rule), we can compute the derivative of the function $f(x) = \tan x$ as follows

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos x \cos x - \sin x (- \sin x)}{\cos^2 x} = \frac{1}{\cos^2 x},$$

for any $x \in \mathbb{R}$ for which $\cos x \neq 0$.

Example 10

To compute the derivative of the function $f(x) = (x^3 + 1)^2$ we can remove the parentheses and then differentiate the result, as follows:

$$f'(x) = \left( (x^3 + 1)^2 \right)' = (x^6 + 2x^3 + 1)' = 6x^5 + 6x^2 = 6x^2 (x^3 + 1).$$

The same derivative can be computed by using the chain rule, as follows:

$$f'(x) = \left( (x^3 + 1)^2 \right)' = 2 (x^3 + 1)^{2-1} \cdot (x^3 + 1)' = 2 (x^3 + 1) \cdot 3x^2 = 6x^2 (x^3 + 1).$$
Example 9

Using the above properties (the quotient rule), we can compute the derivative of the function \( f(x) = \tan x \) as follows

\[
(tan x)' = \left( \frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos x \cos x - \sin x (- \sin x)}{\cos^2 x} = \frac{1}{\cos^2 x},
\]

for any \( x \in \mathbb{R} \) for which \( \cos x \neq 0 \).

Example 10

To compute the derivative of the function \( f(x) = (x^3 + 1)^2 \) we can remove the parentheses and then differentiate the result, as follows:

\[
f'(x) = \left( (x^3 + 1)^2 \right)' = (x^6 + 2x^3 + 1)' = 6x^5 + 6x^2 = 6x^2 (x^3 + 1).
\]

The same derivative can be computed by using the chain rule, as follows:

\[
f'(x) = \left( (x^3 + 1)^2 \right)' = 2 (x^3 + 1)^{2-1} \cdot (x^3 + 1)' = 2 (x^3 + 1) \cdot 3x^2 = 6x^2 (x^3 + 1).
\]
Example 9

Using the above properties (the quotient rule), we can compute the derivative of the function \( f(x) = \tan x \) as follows

\[
(tan x)' = \left( \frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x} = \frac{\cos x \cos x - \sin x (-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x},
\]

for any \( x \in \mathbb{R} \) for which \( \cos x \neq 0 \).

Example 10

To compute the derivative of the function \( f(x) = (x^3 + 1)^2 \) we can remove the parentheses and then differentiate the result, as follows:

\[
f'(x) = \left( (x^3 + 1)^2 \right)' = \left( x^6 + 2x^3 + 1 \right)' = 6x^5 + 6x^2 = 6x^2 (x^3 + 1).
\]

The same derivative can be computed by using the chain rule, as follows:

\[
f'(x) = \left( (x^3 + 1)^2 \right)' = 2 (x^3 + 1)^{2-1} \cdot (x^3 + 1)' = 2 (x^3 + 1) \cdot 3x^2 = 6x^2 (x^3 + 1).
\]
Theorem 11 (Relationship between differentiability and continuity)

If \( f : A \subset \mathbb{R} \to \mathbb{R} \) is differentiable at \( x_0 \) then \( f \) is continuous at \( x_0 \).

Proof.

We will show that \( \lim_{x \to x_0} f(x) = f(x_0) \).

To do this, we write the limit as follows:

\[
\lim_{x \to x_0} f(x) = f(x_0) + \lim_{x \to x_0} (f(x) - f(x_0))
\]

\[
= f(x_0) + \lim_{x \to x_0} \left( \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right)
\]

\[
= f(x_0) + \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0)
\]

\[
= f(x_0) + f'(x_0) \cdot 0
\]

\[
= f(x_0)
\]

where we have used the fact that the last two limits exist and are finite (and therefore the limit of the product is the product of the limits), concluding the proof.
Theorem 11 (Relationship between differentiability and continuity)

If \( f : A \subset \mathbb{R} \to \mathbb{R} \) is differentiable at \( x_0 \) then \( f \) is continuous at \( x_0 \).

Proof.

We will show that \( \lim_{x \to x_0} f(x) = f(x_0) \).

To do this, we write the limit as follows:

\[
\lim_{x \to x_0} f(x) = f(x_0) + \lim_{x \to x_0} (f(x) - f(x_0))
\]

\[
= f(x_0) + \lim_{x \to x_0} \left( \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right)
\]

\[
= f(x_0) + \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0)
\]

\[
= f(x_0) + f'(x_0) \cdot 0
\]

\[
= f(x_0),
\]

where we have used the fact that the last two limits exist and are finite (and therefore the limit of the product is the product of the limits), concluding the proof.
Theorem 11 (Relationship between differentiability and continuity)

If \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) is differentiable at \( x_0 \) then \( f \) is continuous at \( x_0 \).

Proof.

We will show that \( \lim_{x \to x_0} f(x) = f(x_0) \).

To do this, we write the limit as follows:

\[
\lim_{x \to x_0} f(x) = f(x_0) + \lim_{x \to x_0} (f(x) - f(x_0))
\]

\[
= f(x_0) + \lim_{x \to x_0} \left( \frac{f(x) - f(x_0)}{x - x_0} \cdot (x - x_0) \right)
\]

\[
= f(x_0) + \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \cdot \lim_{x \to x_0} (x - x_0)
\]

\[
= f(x_0) + f'(x_0) \cdot 0
\]

\[
= f(x_0)
\]

where we have used the fact that the last two limits exist and are finite (and therefore the limit of the product is the product of the limits), concluding the proof.
As we will see, the derivative of a function is useful for finding the maximum /minimum value of a function.

**Definition 12**

Let \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) and let \( x_0 \in A \). We say that \( x_0 \) is a:

i) **local / relative minimum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon)
\]

(5)

ii) **local / relative maximum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon)
\]

(6)

iii) **local / relative extremum point** for \( f \) if \( x_0 \) is either a local minimum or a local maximum point for \( f \).

If we have strict inequalities in (5) or (6), we say that \( x_0 \) is a **strict local minimum** / **strict local maximum** / **strict local extremum** for \( f \).

If the inequalities in (5) or (6) hold for any \( x \in A \), then \( x_0 \) is called an **absolute(global) minimum** / **maximum** / **extremum point** for \( f \).
As we will see, the derivative of a function is useful for finding the maximum /minimum value of a function.

**Definition 12**

Let $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$ and let $x_0 \in A$. We say that $x_0$ is a:

i) **local / relative minimum point** for $f$ if for some $\varepsilon > 0$ we have

$$f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon)$$ (5)

ii) **local / relative maximum point** for $f$ if for some $\varepsilon > 0$ we have

$$f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon)$$ (6)

iii) **local / relative extremum point** for $f$ if $x_0$ is either a local minimum or a local maximum point for $f$.

If we have strict inequalities in (5) or (6), we say that $x_0$ is a **strict local minimum / strict local maximum / strict local extremum** for $f$.

If the inequalities in (5) or (6) hold for any $x \in A$, then $x_0$ is called an **absolute(global) minimum / maximum / extremum point** for $f$.
As we will see, the derivative of a function is useful for finding the maximum/minimum value of a function.

**Definition 12**

Let \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) and let \( x_0 \in A \). We say that \( x_0 \) is a:

i) **local / relative minimum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
    f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon)
\]  \( (5) \)

ii) **local / relative maximum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
    f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon)
\]  \( (6) \)

iii) **local / relative extremum point** for \( f \) if \( x_0 \) is either a local minimum or a local maximum point for \( f \).

If we have strict inequalities in (5) or (6), we say that \( x_0 \) is a **strict local minimum** / **strict local maximum** / **strict local extremum** for \( f \).

If the inequalities in (5) or (6) hold for any \( x \in A \), then \( x_0 \) is called an **absolute (global) minimum** / **maximum** / **extremum point** for \( f \).
As we will see, the derivative of a function is useful for finding the maximum /minimum value of a function.

**Definition 12**

Let \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) and let \( x_0 \in A \). We say that \( x_0 \) is a:

i) **local / relative minimum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
  f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon) \tag{5}
\]

ii) **local / relative maximum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
  f(x_0) \geq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon) \tag{6}
\]

iii) **local / relative extremum point** for \( f \) if \( x_0 \) is either a local minimum or a local maximum point for \( f \).

If we have strict inequalities in (5) or (6), we say that \( x_0 \) is a **strict local minimum** / **strict local maximum** / **strict local extremum** for \( f \).

If the inequalities in (5) or (6) hold for any \( x \in A \), then \( x_0 \) is called an **absolute (global) minimum** / **maximum** / **extremum point** for \( f \).
As we will see, the derivative of a function is useful for finding the maximum/minimum value of a function.

**Definition 12**

Let \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) and let \( x_0 \in A \). We say that \( x_0 \) is a:

i) **local / relative minimum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon) \tag{5}
\]

ii) **local / relative maximum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon) \tag{6}
\]

iii) **local / relative extremum point** for \( f \) if \( x_0 \) is either a local minimum or a local maximum point for \( f \).

If we have strict inequalities in (5) or (6), we say that \( x_0 \) is a **strict local minimum / strict local maximum / strict local extremum** for \( f \).

If the inequalities in (5) or (6) hold for any \( x \in A \), then \( x_0 \) is called an **absolute (global) minimum / maximum / extremum point** for \( f \).
As we will see, the derivative of a function is useful for finding the maximum /minimum value of a function.

**Definition 12**

Let \( f : A \subset \mathbb{R} \to \mathbb{R} \) and let \( x_0 \in A \). We say that \( x_0 \) is a:

i) **local / relative minimum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon)
\]  

(5)

ii) **local / relative maximum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon)
\]  

(6)

iii) **local / relative extremum point** for \( f \) if \( x_0 \) is either a local minimum or a local maximum point for \( f \).

If we have strict inequalities in (5) or (6), we say that \( x_0 \) is a **strict** local minimum / strict local maximum / strict local extremum for \( f \).

If the inequalities in (5) or (6) hold for any \( x \in A \), then \( x_0 \) is called an absolute(global) minimum / maximum / extremum point for \( f \).
As we will see, the derivative of a function is useful for finding the maximum /minimum value of a function.

**Definition 12**

Let \( f : A \subset \mathbb{R} \to \mathbb{R} \) and let \( x_0 \in A \). We say that \( x_0 \) is a:

i) **local / relative minimum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon)
\]  

(5)

ii) **local / relative maximum point** for \( f \) if for some \( \varepsilon > 0 \) we have

\[
f(x_0) \leq f(x), \quad x \in A \cap (x_0 - \varepsilon, x_0 + \varepsilon)
\]  

(6)

iii) **local / relative extremum point** for \( f \) if \( x_0 \) is either a local minimum or a local maximum point for \( f \).

If we have strict inequalities in (5) or (6), we say that \( x_0 \) is a **strict** local minimum / **strict** local maximum / **strict** local extremum for \( f \).

If the inequalities in (5) or (6) hold for any \( x \in A \), then \( x_0 \) is called an **absolute**(global) minimum / maximum / extremum point for \( f \).
Example 13

Consider the function $f : [-2, 3] \to \mathbb{R}$ having the graph below. We see that $x = -2$, $x = -0, 5$ and $x = 2$ are relative minimum points of $f$, $x = -1, 5$, $x = 1$ and $x = 3$ are relative maximum points of $f$, $x = 3$ is the absolute minimum point for $f$ and $x = 3$ is the absolute maximum point for $f$.

**Figure**: The relative extremum points of the function $f : [-2, 3] \to \mathbb{R}$. 
Theorem 14 (Fermat’s theorem)

If $f : A \subset \mathbb{R} \to \mathbb{R}$ has a relative extremum at an interior point $x_0$ of $A$ and $f$ is differentiable at $x_0$ then $f'(x_0) = 0$.

Remark 15

In this above theorem it is essential that:
- $x_0$ is an interior point of $A$ (the domain of definition of the function $f$)
- $f$ is differentiable at $x_0$ (if $f$ is not differentiable at $x_0$, then $f'(x_0)$ is not defined, so it cannot equal 0!).

To see this, consider the function $f : [1, 2] \to \mathbb{R}$ defined by $f(x) = x$. Note that $x_0 = 1$ is a relative minimum point for $f$ and $x_0 = 2$ is a relative maximum point for $f$. We see that $f$ is differentiable on $[1, 2]$ and $f'(x) = 1$ for any $x \in [1, 2]$.

In particular, $f'(1) = f'(2) = 1 \neq 0$, which shows that the conclusion in Fermat’s theorem does not hold (note that $x_0 = 1$ and $x_0 = 2$ are not interior points of the domain $[1, 2]$ of $f$).
Theorem 14 (Fermat’s theorem)

If \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) has a relative extremum at an interior point \( x_0 \) of \( A \) and \( f \) is differentiable at \( x_0 \) then \( f'(x_0) = 0 \).

Remark 15

In this above theorem it is essential that:

- \( x_0 \) is an interior point of \( A \) (the domain of definition of the function \( f \))
- \( f \) is differentiable at \( x_0 \) (if \( f \) is not differentiable at \( x_0 \), then \( f'(x_0) \) is not defined, so it cannot equal 0!).

To see this, consider the function \( f : [1, 2] \rightarrow \mathbb{R} \) defined by \( f(x) = x \).
Note that \( x_0 = 1 \) is a relative minimum point for \( f \) and \( x_0 = 2 \) is a relative maximum point for \( f \).
We see that \( f \) is differentiable on \([1, 2]\) and \( f'(x) = 1 \) for any \( x \in [1, 2] \).
In particular, \( f'(1) = f'(2) = 1 \neq 0 \), which shows that the conclusion in Fermat’s theorem does not hold (note that \( x_0 = 1 \) and \( x_0 = 2 \) are not interior points of the domain \([1, 2]\) of \( f \)).
Theorem 14 (Fermat’s theorem)

If \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) has a relative extremum at an interior point \( x_0 \) of \( A \) and \( f \) is differentiable at \( x_0 \) then \( f' (x_0) = 0 \).

Remark 15

In this above theorem it is essential that:

- \( x_0 \) is an interior point of \( A \) (the domain of definition of the function \( f \))
- \( f \) is differentiable at \( x_0 \) (if \( f \) is not differentiable at \( x_0 \), then \( f' (x_0) \) is not defined, so it cannot equal 0!).

To see this, consider the function \( f : [1, 2] \rightarrow \mathbb{R} \) defined by \( f (x) = x \). Note that \( x_0 = 1 \) is a relative minimum point for \( f \) and \( x_0 = 2 \) is a relative maximum point for \( f \).

We see that \( f \) is differentiable on \([1, 2]\) and \( f' (x) = 1 \) for any \( x \in [1, 2] \).

In particular, \( f' (1) = f' (2) = 1 \neq 0 \), which shows that the conclusion in Fermat’s theorem does not hold (note that \( x_0 = 1 \) and \( x_0 = 2 \) are not interior points of the domain \([1, 2]\) of \( f \)).
Theorem 14 (Fermat’s theorem)

If \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) has a relative extremum at an interior point \( x_0 \) of \( A \) and \( f \) is differentiable at \( x_0 \) then \( f' (x_0) = 0 \).

Remark 15

In this above theorem it is essential that:

- \( x_0 \) is an interior point of \( A \) (the domain of definition of the function \( f \))
- \( f \) is differentiable at \( x_0 \) (if \( f \) is not differentiable at \( x_0 \), then \( f' (x_0) \) is not defined, so it cannot equal 0!).

To see this, consider the function \( f : [1, 2] \rightarrow \mathbb{R} \) defined by \( f (x) = x \). Note that \( x_0 = 1 \) is a relative minimum point for \( f \) and \( x_0 = 2 \) is a relative maximum point for \( f \).

We see that \( f \) is differentiable on \([1, 2]\) and \( f' (x) = 1 \) for any \( x \in [1, 2] \).

In particular, \( f' (1) = f' (2) = 1 \neq 0 \), which shows that the conclusion in Fermat’s theorem does not hold (note that \( x_0 = 1 \) and \( x_0 = 2 \) are not interior points of the domain \([1, 2]\) of \( f \)).
Theorem 14 (Fermat’s theorem)

If \( f : A \subset \mathbb{R} \to \mathbb{R} \) has a relative extremum at an interior point \( x_0 \) of \( A \) and \( f \) is differentiable at \( x_0 \) then \( f' (x_0) = 0 \).

Remark 15

In this above theorem it is essential that:
- \( x_0 \) is an interior point of \( A \) (the domain of definition of the function \( f \))
- \( f \) is differentiable at \( x_0 \) (if \( f \) is not differentiable at \( x_0 \), then \( f' (x_0) \) is not defined, so it cannot equal 0!).

To see this, consider the function \( f : [1, 2] \to \mathbb{R} \) defined by \( f (x) = x \). Note that \( x_0 = 1 \) is a relative minimum point for \( f \) and \( x_0 = 2 \) is a relative maximum point for \( f \).
We see that \( f \) is differentiable on \([1, 2]\) and \( f' (x) = 1 \) for any \( x \in [1, 2] \).
In particular, \( f' (1) = f' (2) = 1 \neq 0 \), which shows that the conclusion in Fermat’s theorem does not hold (note that \( x_0 = 1 \) and \( x_0 = 2 \) are not interior points of the domain \([1, 2]\) of \( f \)).
Theorem 16 (Rolle’s theorem)

Let \( f : [a, b] \rightarrow \mathbb{R} \) and assume that \( f \) is continuous on \([a, b]\), differentiable on \((a, b)\) and \( f(a) = f(b) \). Then there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

Proof.

Since \( f \) is continuous on \([a, b]\), by Weierstrass boundedness theorem, \( f \) is bounded and attains its bounds on \([a, b]\): there exist \( x_m, x_M \in [a, b] \) such that

\[
f(x_m) \leq f(x) \leq f(x_M), \quad x \in [a, b].
\]

If \( f(x_m) = f(x_M) \) then \( f \) is constant on \([a, b]\), so \( f' \) is identically zero on \([a, b]\), and we can choose any \( c \in (a, b) \) in this case.

If \( f(x_m) \neq f(x_M) \), then \( x_m, x_M \) cannot be both endpoints of \([a, b]\) (since by hypothesis \( f(a) = f(b) \)). Therefore at least one of \( x_m \) or \( x_M \) is an interior point of \([a, b]\), say \( x_m \) is an interior point of \([a, b]\).

By Fermat’s theorem it follows that \( f'(x_m) = 0 \), so we can choose \( c = x_m \in (a, b) \) and conclude the proof.
Theorem 16 (Rolle’s theorem)

Let $f : [a, b] \to \mathbb{R}$ and assume that $f$ is continuous on $[a, b]$, differentiable on $(a, b)$ and $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof.

Since $f$ is continuous on $[a, b]$, by Weierstrass boundedness theorem, $f$ is bounded and attains its bounds on $[a, b]$: there exist $x_m, x_M \in [a, b]$ such that

$$f(x_m) \leq f(x) \leq f(x_M), \quad x \in [a, b].$$

If $f(x_m) = f(x_M)$ then $f$ is constant on $[a, b]$, so $f'$ is identically zero on $[a, b]$, and we can choose any $c \in (a, b)$ in this case.

If $f(x_m) \neq f(x_M)$, then $x_m, x_M$ cannot be both endpoints of $[a, b]$ (since by hypothesis $f(a) = f(b)$). Therefore at least one of $x_m$ or $x_M$ is an interior point of $[a, b]$, say $x_m$ is an interior point of $[a, b]$.

By Fermat’s theorem it follows that $f'(x_m) = 0$, so we can choose $c = x_m \in (a, b)$ and conclude the proof.
Theorem 16 (Rolle’s theorem)

Let \( f : [a, b] \to \mathbb{R} \) and assume that \( f \) is continuous on \([a, b]\), differentiable on \((a, b)\) and \( f(a) = f(b) \). Then there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

Proof.

Since \( f \) is continuous on \([a, b]\), by Weierstrass boundedness theorem, \( f \) is bounded and attains its bounds on \([a, b]\): there exist \( x_m, x_M \in [a, b] \) such that

\[
f(x_m) \leq f(x) \leq f(x_M), \quad x \in [a, b].
\]

If \( f(x_m) = f(x_M) \) then \( f \) is constant on \([a, b]\), so \( f' \) is identically zero on \([a, b]\), and we can choose any \( c \in (a, b) \) in this case.

If \( f(x_m) \neq f(x_M) \), then \( x_m, x_M \) cannot be both endpoints of \([a, b]\) (since by hypothesis \( f(a) = f(b) \)). Therefore at least one of \( x_m \) or \( x_M \) is an interior point of \([a, b]\), say \( x_m \) is an interior point of \([a, b]\).

By Fermat’s theorem it follows that \( f'(x_m) = 0 \), so we can choose \( c = x_m \in (a, b) \) and conclude the proof.
Theorem 16 (Rolle’s theorem)

Let \( f : [a, b] \to \mathbb{R} \) and assume that \( f \) is continuous on \([a, b]\), differentiable on \((a, b)\) and \( f(a) = f(b) \). Then there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

Proof.

Since \( f \) is continuous on \([a, b]\), by Weierstrass boundedness theorem, \( f \) is bounded and attains its bounds on \([a, b]\): there exist \( x_m, x_M \in [a, b] \) such that

\[
f(x_m) \leq f(x) \leq f(x_M), \quad x \in [a, b].
\]

If \( f(x_m) = f(x_M) \) then \( f \) is constant on \([a, b]\), so \( f' \) is identically zero on \([a, b]\), and we can choose any \( c \in (a, b) \) in this case.

If \( f(x_m) \neq f(x_M) \), then \( x_m, x_M \) cannot be both endpoints of \([a, b]\) (since by hypothesis \( f(a) = f(b) \)). Therefore at least one of \( x_m \) or \( x_M \) is an interior point of \([a, b]\), say \( x_m \) is an interior point of \([a, b]\).

By Fermat’s theorem it follows that \( f'(x_m) = 0 \), so we can choose \( c = x_m \in (a, b) \) and conclude the proof.
Theorem 16 (Rolle’s theorem)

Let \( f : [a, b] \rightarrow \mathbb{R} \) and assume that \( f \) is continuous on \([a, b]\), differentiable on \((a, b)\) and \( f(a) = f(b) \). Then there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

Proof.

Since \( f \) is continuous on \([a, b]\), by Weierstrass boundedness theorem, \( f \) is bounded and attains its bounds on \([a, b]\): there exist \( x_m, x_M \in [a, b] \) such that

\[
    f(x_m) \leq f(x) \leq f(x_M), \quad x \in [a, b].
\]

If \( f(x_m) = f(x_M) \) then \( f \) is constant on \([a, b]\), so \( f' \) is identically zero on \([a, b]\), and we can choose any \( c \in (a, b) \) in this case.

If \( f(x_m) \neq f(x_M) \), then \( x_m, x_M \) cannot be both endpoints of \([a, b]\) (since by hypothesis \( f(a) = f(b) \)). Therefore at least one of \( x_m \) or \( x_M \) is an interior point of \([a, b]\), say \( x_m \) is an interior point of \([a, b]\).

By Fermat’s theorem it follows that \( f'(x_m) = 0 \), so we can choose \( c = x_m \in (a, b) \) and conclude the proof.
**Theorem 16 (Rolle’s theorem)**

Let \( f : [a, b] \to \mathbb{R} \) and assume that \( f \) is continuous on \([a, b]\), differentiable on \((a, b)\) and \( f(a) = f(b) \). Then there exists \( c \in (a, b) \) such that \( f'(c) = 0 \).

**Proof.**

Since \( f \) is continuous on \([a, b]\), by Weierstrass boundedness theorem, \( f \) is bounded and attains its bounds on \([a, b]\): there exist \( x_m, x_M \in [a, b] \) such that

\[
 f(x_m) \leq f(x) \leq f(x_M), \quad x \in [a, b].
\]

If \( f(x_m) = f(x_M) \) then \( f \) is constant on \([a, b]\), so \( f' \) is identically zero on \([a, b]\), and we can choose any \( c \in (a, b) \) in this case.

If \( f(x_m) \neq f(x_M) \), then \( x_m, x_M \) cannot be both endpoints of \([a, b]\) (since by hypothesis \( f(a) = f(b) \)). Therefore at least one of \( x_m \) or \( x_M \) is an interior point of \([a, b]\), say \( x_m \) is an interior point of \([a, b]\).

By Fermat’s theorem it follows that \( f'(x_m) = 0 \), so we can choose \( c = x_m \in (a, b) \) and conclude the proof.
Theorem 17 (Lagrange’s theorem)

Let \( f : [a, b] \rightarrow \mathbb{R} \) and assume that \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there exists \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]  

(7)

Proof.

Apply Rolle’s theorem to the function \( F : [a, b] \rightarrow \mathbb{R} \) defined by

\[
F(x) = f(x) - \frac{f(b) - f(a)}{b - a} (x - a).
\]
Theorem 17 (Lagrange’s theorem)

Let \( f : [a, b] \to \mathbb{R} \) and assume that \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Then there exists \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\] (7)

Proof.

Apply Rolle’s theorem to the function \( F : [a, b] \to \mathbb{R} \) defined by

\[
F(x) = f(x) - \frac{f(b) - f(a)}{b - a} (x - a).
\]
Proposition 18

Assume \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable \((a, b)\).

i) If \( f'(x) \geq 0 \) for \( x \in (a, b) \) then \( f \) is increasing on \([a, b]\);

ii) If \( f'(x) \leq 0 \) for \( x \in (a, b) \) then \( f \) is decreasing on \([a, b]\).

iii) If \( f'(x) = 0 \) for \( x \in (a, b) \) then \( f \) is constant on \([a, b]\).

Example 19

Consider the function \( f(x) = x^3 \).

\( f \) is continuous and differentiable on \( \mathbb{R} \), and \( f'(x) = 3x^2 \geq 0 \) for any \( x \in \mathbb{R} \).

By the previous proposition it follows that the function \( f(x) = x^3 \) is increasing on \( \mathbb{R} \) (it is in fact strictly increasing).
Proposition 18

Assume \( f : [a, b] \rightarrow \mathbb{R} \) is continuous on \([a, b]\) and differentiable \((a, b)\).

i) If \( f'(x) \geq 0 \) for \( x \in (a, b) \) then \( f \) is increasing on \([a, b]\);

ii) If \( f'(x) \leq 0 \) for \( x \in (a, b) \) then \( f \) is decreasing on \([a, b]\).

iii) If \( f'(x) = 0 \) for \( x \in (a, b) \) then \( f \) is constant on \([a, b]\).

Example 19

Consider the function \( f(x) = x^3 \).

\( f \) is continuous and differentiable on \( \mathbb{R} \), and \( f'(x) = 3x^2 \geq 0 \) for any \( x \in \mathbb{R} \).

By the previous proposition it follows that the function \( f(x) = x^3 \) is increasing on \( \mathbb{R} \) (it is in fact strictly increasing).
Proposition 18

Assume \( f : [a, b] \rightarrow \mathbb{R} \) is continuous on \([a, b]\) and differentiable \((a, b)\).

i) If \( f'(x) \geq 0 \) for \( x \in (a, b) \) then \( f \) is increasing on \([a, b]\);

ii) If \( f'(x) \leq 0 \) for \( x \in (a, b) \) then \( f \) is decreasing on \([a, b]\).

iii) If \( f'(x) = 0 \) for \( x \in (a, b) \) then \( f \) is constant on \([a, b]\).

Example 19

Consider the function \( f(x) = x^3 \).

\( f \) is continuous and differentiable on \( \mathbb{R} \), and \( f'(x) = 3x^2 \geq 0 \) for any \( x \in \mathbb{R} \).

By the previous proposition it follows that the function \( f(x) = x^3 \) is increasing on \( \mathbb{R} \) (it is in fact strictly increasing).
Proposition 18

Assume \( f : [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable \((a, b)\).

i) If \( f'(x) \geq 0 \) for \( x \in (a, b) \) then \( f \) is increasing on \([a, b]\);

ii) If \( f'(x) \leq 0 \) for \( x \in (a, b) \) then \( f \) is decreasing on \([a, b]\).

iii) If \( f'(x) = 0 \) for \( x \in (a, b) \) then \( f \) is constant on \([a, b]\).

Example 19

Consider the function \( f(x) = x^3 \).

\( f \) is continuous and differentiable on \( \mathbb{R} \), and \( f'(x) = 3x^2 \geq 0 \) for any \( x \in \mathbb{R} \).

By the previous proposition it follows that the function \( f(x) = x^3 \) is increasing on \( \mathbb{R} \) (it is in fact strictly increasing).
Proposition 18

Assume \( f: [a, b] \to \mathbb{R} \) is continuous on \([a, b]\) and differentiable \((a, b)\).

i) If \( f'(x) \geq 0 \) for \( x \in (a, b) \) then \( f \) is increasing on \([a, b]\);

ii) If \( f'(x) \leq 0 \) for \( x \in (a, b) \) then \( f \) is decreasing on \([a, b]\).

iii) If \( f'(x) = 0 \) for \( x \in (a, b) \) then \( f \) is constant on \([a, b]\).

Example 19

Consider the function \( f(x) = x^3 \).

\( f \) is continuous and differentiable on \( \mathbb{R} \), and \( f'(x) = 3x^2 \geq 0 \) for any \( x \in \mathbb{R} \).

By the previous proposition it follows that the function \( f(x) = x^3 \) is increasing on \( \mathbb{R} \) (it is in fact strictly increasing).
Proposition 18

Assume $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable $(a, b)$.

i) If $f'(x) \geq 0$ for $x \in (a, b)$ then $f$ is increasing on $[a, b]$;

ii) If $f'(x) \leq 0$ for $x \in (a, b)$ then $f$ is decreasing on $[a, b]$.

iii) If $f'(x) = 0$ for $x \in (a, b)$ then $f$ is constant on $[a, b]$.

Example 19

Consider the function $f(x) = x^3$.

$f$ is continuous and differentiable on $\mathbb{R}$, and $f'(x) = 3x^2 \geq 0$ for any $x \in \mathbb{R}$.

By the previous proposition it follows that the function $f(x) = x^3$ is increasing on $\mathbb{R}$ (it is in fact strictly increasing).
Theorem 20 (Cauchy’s theorem)

Let \( f, g : [a, b] \to \mathbb{R} \) and assume that \( f \) and \( g \) are continuous on \([a, b]\), differentiable on \((a, b)\) and \( g'(x) \neq 0 \) for \( x \in (a, b) \).

Then \( g(b) - g(a) \neq 0 \) and there exists \( c \in (a, b) \) such that

\[
\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.
\]

(8)

Proof.

Apply Rolle’s theorem to the function \( F : [a, b] \to \mathbb{R} \) defined by

\[
F(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} \left( g(x) - g(a) \right).
\]
Theorem 20 (Cauchy’s theorem)

Let \( f, g : [a, b] \rightarrow \mathbb{R} \) and assume that \( f \) and \( g \) are continuous on \([a, b]\), differentiable on \((a, b)\) and \( g'(x) \neq 0 \) for \( x \in (a, b) \).

Then \( g(b) - g(a) \neq 0 \) and there exists \( c \in (a, b) \) such that

\[
\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.
\]

(8)

Proof.

Apply Rolle’s theorem to the function \( F : [a, b] \rightarrow \mathbb{R} \) defined by

\[
F(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a)).
\]
Theorem 20 (Cauchy’s theorem)

Let \( f, g : [a, b] \rightarrow \mathbb{R} \) and assume that \( f \) and \( g \) are continuous on \([a, b]\), differentiable on \((a, b)\) and \( g'(x) \neq 0 \) for \( x \in (a, b) \).
Then \( g(b) - g(a) \neq 0 \) and there exists \( c \in (a, b) \) such that

\[
\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.
\]

(8)

Proof.

Apply Rolle’s theorem to the function \( F : [a, b] \rightarrow \mathbb{R} \) defined by

\[
F(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a)).
\]
Theorem 21 (L’Hôpital’s rule)

Let \( f, g : [a, b] \to \mathbb{R} \) and \( x_0 \in [a, b] \). If \( f \) and \( g \) are continuous on \( [a, b] \) and differentiable on \((a, b) \setminus \{x_0\}\), \( g' (x) \neq 0 \) for \( x \in (a, b) \setminus \{x_0\} \) and \( f (x_0) = g (x_0) = 0 \), then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)},
\]

provided the last limit exists.

Proof.

Idea: apply Cauchy’s theorem

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{c \to x_0} \frac{f'(c)}{g'(c)}.
\]

Remark 22 (Extensions’s of L’Hôpital’s rule)

- L’Hôpital’s rule holds for sided limits (left / right limits at a point).
- It also holds in the case when \( x_0 = \pm \infty \).
- It can also be applied for limits of the form \( \frac{\pm \infty}{\pm \infty} \).
Theorem 21 (L’Hôpital’s rule)

Let \( f, g : [a, b] \to \mathbb{R} \) and \( x_0 \in [a, b] \). If \( f \) and \( g \) are continuous on \([a, b]\) and differentiable on \((a, b) - \{x_0\}\), \( g' (x) \neq 0 \) for \( x \in (a, b) - \{x_0\} \) and \( f(x_0) = g(x_0) = 0 \), then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} ,
\]

provided the last limit exists.

Proof.

Idea: apply Cauchy’s theorem

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{c \to x_0} \frac{f'(c)}{g'(c)} .
\]

Remark 22 (Extensions’s of L’Hôpital’s rule)

- L’Hôpital’s rule holds for sided limits (left / right limits at a point).
- It also holds in the case when \( x_0 = \pm \infty \).
- It can also be applied for limits of the form \( \frac{\pm \infty}{\pm \infty} \).
Theorem 21 (L’Hôpital’s rule)

Let \( f, g : [a, b] \rightarrow \mathbb{R} \) and \( x_0 \in [a, b] \). If \( f \) and \( g \) are continuous on \([a, b]\) and differentiable on \((a, b) - \{x_0\}\), \( g' (x) \neq 0 \) for \( x \in (a, b) - \{x_0\} \) and \( f(x_0) = g(x_0) = 0 \), then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)},
\]

(9)

provided the last limit exists.

Proof.

**Idea:** apply Cauchy’s theorem

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{c \to x_0} \frac{f'(c)}{g'(c)}.
\]

Remark 22 (Extensions’s of L’Hôpital’s rule)

- L’Hôpital’s rule holds for sided limits (left / right limits at a point).
- It also holds in the case when \( x_0 = \pm \infty \).
- It can also be applied for limits of the form \( \pm \infty \) / \( \pm \infty \).
Theorem 21 (L’Hôpital’s rule)

Let \( f, g : [a, b] \to \mathbb{R} \) and \( x_0 \in [a, b] \). If \( f \) and \( g \) are continuous on \([a, b]\) and differentiable on \((a, b) - \{x_0\}\), \( g'(x) \neq 0 \) for \( x \in (a, b) - \{x_0\} \) and \( f(x_0) = g(x_0) = 0 \), then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)},
\]

provided the last limit exists.

Proof.

**Idea:** apply Cauchy’s theorem

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{c \to x_0} \frac{f'(c)}{g'(c)}.
\]

Remark 22 (Extensions’s of L’Hôpital’s rule)

- L’Hôpital’s rule holds for sided limits (left / right limits at a point).
- It also holds in the case when \( x_0 = \pm \infty \).
- It can also be applied for limits of the form \( \pm \infty / \pm \infty \).
Theorem 21 (L’Hôpital’s rule)

Let \( f, g : [a, b] \rightarrow \mathbb{R} \) and \( x_0 \in [a, b] \). If \( f \) and \( g \) are continuous on \([a, b]\) and differentiable on \((a, b) - \{x_0\}\), \( g' (x) \neq 0 \) for \( x \in (a, b) - \{x_0\} \) and \( f (x_0) = g (x_0) = 0 \), then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)},
\]

provided the last limit exists.

Proof.

Idea: apply Cauchy’s theorem

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{c \to x_0} \frac{f'(c)}{g'(c)}.
\]

Remark 22 (Extensions’s of L’Hôpital’s rule)

- L’Hôpital’s rule holds for sided limits (left / right limits at a point).
- It also holds in the case when \( x_0 = \pm \infty \).
- It can also be applied for limits of the form \( \pm \infty / \pm \infty \).
Theorem 21 (L’Hôpital’s rule)

Let \( f, g : [a, b] \to \mathbb{R} \) and \( x_0 \in [a, b] \). If \( f \) and \( g \) are continuous on \([a, b]\) and differentiable on \((a, b) - \{x_0\}\), \( g'(x) \neq 0 \) for \( x \in (a, b) - \{x_0\} \) and \( f(x_0) = g(x_0) = 0 \), then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)},
\]  

(9)

provided the last limit exists.

Proof.

Idea: apply Cauchy’s theorem

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{c \to x_0} \frac{f'(c)}{g'(c)}.
\]

Remark 22 (Extensions’s of L’Hôpital’s rule)

- L’Hôpital’s rule holds for sided limits (left / right limits at a point).
- It also holds in the case when \( x_0 = \pm \infty \).
- It can also be applied for limits of the form \( \pm \frac{\infty}{\pm \infty} \).
Theorem 21 (L’Hôpital’s rule)

Let \( f, g : [a, b] \to \mathbb{R} \) and \( x_0 \in [a, b] \). If \( f \) and \( g \) are continuous on \([a, b]\) and differentiable on \((a, b) - \{x_0\}\), \( g'(x) \neq 0 \) for \( x \in (a, b) - \{x_0\} \) and \( f(x_0) = g(x_0) = 0 \), then

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)},
\]

(9)

provided the last limit exists.

Proof.

Idea: apply Cauchy’s theorem

\[
\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \lim_{c \to x_0} \frac{f'(c)}{g'(c)}.
\]

Remark 22 (Extensions’s of L’Hôpital’s rule)

- L’Hôpital’s rule holds for sided limits (left / right limits at a point).
- It also holds in the case when \( x_0 = \pm \infty \).
- It can also be applied for limits of the form \( \pm \infty \).
Example 23

As an example, let us compute the limit

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2}.$$ 

We see that the limit is of the type $0/0$ and the functions involved satisfy the hypotheses of L’Hôpital’s rule above.

Applying the theorem we obtain

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2} = \lim_{x \to 0} \frac{(1 + x - e^x)'}{(x^2)'} = \lim_{x \to 0} \frac{1 - e^x}{2x}.$$

Applying again the theorem (the limit is again of the type $0/0$), we obtain

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2} = \lim_{x \to 0} \frac{1 - e^x}{2x} = \lim_{x \to 0} \frac{(1 - e^x)'}{(2x)'} = \lim_{x \to 0} \frac{-e^x}{2} = -\frac{1}{2}.$$
Example 23

As an example, let us compute the limit

\[ \lim_{x \to 0} \frac{1 + x - e^x}{x^2}. \]

We see that the limit is of the type \( \frac{0}{0} \) and the functions involved satisfy the hypotheses of L’Hôpital’s rule above.

Applying the theorem we obtain

\[ \lim_{x \to 0} \frac{1 + x - e^x}{x^2} = \lim_{x \to 0} \frac{(1 + x - e^x)'}{(x^2)'} = \lim_{x \to 0} \frac{1 - e^x}{2x}. \]

Applying again the theorem (the limit is again of the type \( \frac{0}{0} \)), we obtain

\[ \lim_{x \to 0} \frac{1 + x - e^x}{x^2} = \lim_{x \to 0} \frac{1 - e^x}{2x} = \lim_{x \to 0} \frac{(1 - e^x)'}{(2x)'} = \lim_{x \to 0} \frac{-e^x}{2} = -\frac{1}{2}. \]
Example 23

As an example, let us compute the limit

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2}.$$

We see that the limit is of the type $\frac{0}{0}$ and the functions involved satisfy the hypotheses of L’Hôpital’s rule above. Applying the theorem we obtain

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2} = \lim_{x \to 0} \frac{(1 + x - e^x)^'}{(x^2)^'} = \lim_{x \to 0} \frac{1 - e^x}{2x}.$$

Applying again the theorem (the limit is again of the type $\frac{0}{0}$), we obtain

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2} = \lim_{x \to 0} \frac{1 - e^x}{2x} = \lim_{x \to 0} \frac{(1 - e^x)^'}{(2x)^'} = \lim_{x \to 0} \frac{-e^x}{2} = -\frac{1}{2}.$$
Example 23

As an example, let us compute the limit

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2}.$$ 

We see that the limit is of the type $\frac{0}{0}$ and the functions involved satisfy the hypotheses of L’Hôpital’s rule above.

Applying the theorem we obtain

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2} = \lim_{x \to 0} \frac{(1 + x - e^x)'}{(x^2)'} = \lim_{x \to 0} \frac{1 - e^x}{2x}.$$ 

Applying again the theorem (the limit is again of the type $\frac{0}{0}$), we obtain

$$\lim_{x \to 0} \frac{1 + x - e^x}{x^2} = \lim_{x \to 0} \frac{1 - e^x}{2} = \lim_{x \to 0} \frac{-e^x}{2} = -\frac{1}{2}.$$
**Remark:** when applying L’Hôpital’s rule, it is essential to check that the limit “on the right” exists, otherwise applying L’Hôpital’s rule may lead to erroneous conclusions, as shown in the example below.

**Example 24**

Consider the limit \( \lim_{x \to \infty} \frac{x + \sin x}{x} \). It is not difficult to see that

\[
\frac{x - 1}{x} \leq \frac{x + \sin x}{x} \leq \frac{x + 1}{x}, \quad x > 0,
\]

and therefore

\[
1 = \lim_{x \to \infty} \frac{x - 1}{x} \leq \lim_{x \to \infty} \frac{x + \sin x}{x} \leq \lim_{x \to \infty} \frac{x + 1}{x} = 1,
\]

which shows that

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = 1.
\]

**Applying incorrectly** L’Hôpital’s rule we obtain

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \frac{(x + \sin x)'}{(x)'} = \lim_{x \to \infty} \frac{1 + \cos x}{1} = \lim_{x \to \infty} (1 + \cos x) - \text{DNE},
\]

which “shows” that the given limit does not exist (we proved above that the limit is in fact 1).
**Remark:** when applying L’Hôpital’s rule, it is essential to check that the limit “on the right” exists, otherwise applying L’Hôpital’s rule may lead to erroneous conclusions, as shown in the example below.

**Example 24**

Consider the limit \( \lim_{x \to \infty} \frac{x + \sin x}{x} \). It is not difficult to see that

\[
\frac{x - 1}{x} \leq \frac{x + \sin x}{x} \leq \frac{x + 1}{x}, \quad x > 0,
\]

and therefore

\[
1 = \lim_{x \to \infty} \frac{x - 1}{x} \leq \lim_{x \to \infty} \frac{x + \sin x}{x} \leq \lim_{x \to \infty} \frac{x + 1}{x} = 1,
\]

which shows that

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = 1.
\]

**Applying incorrectly** L’Hôpital’s rule we obtain

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \frac{(x + \sin x)'}{(x)'} = \lim_{x \to \infty} \frac{1 + \cos x}{1} = \lim_{x \to \infty} (1 + \cos x) - \text{DNE},
\]

which “shows” that the given limit does not exist (we proved above that the limit is in fact 1).
**Remark:** when applying L’Hôpital’s rule, it is essential to check that the limit “on the right” exists, otherwise applying L’Hôpital’s rule may lead to erroneous conclusions, as shown in the example below.

**Example 24**

Consider the limit \( \lim_{x \to \infty} \frac{x + \sin x}{x} \). It is not difficult to see that

\[
\frac{x - 1}{x} \leq \frac{x + \sin x}{x} \leq \frac{x + 1}{x}, \quad x > 0,
\]

and therefore

\[
1 = \lim_{x \to \infty} \frac{x - 1}{x} \leq \lim_{x \to \infty} \frac{x + \sin x}{x} \leq \lim_{x \to \infty} \frac{x + 1}{x} = 1,
\]

which shows that

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = 1.
\]

Applying **incorrectly** L’Hôpital’s rule we obtain

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \frac{(x + \sin x)'}{(x)'} = \lim_{x \to \infty} \frac{1 + \cos x}{1} = \lim_{x \to \infty} (1 + \cos x) - \text{DNE},
\]

which “shows” that the given limit does not exist (we proved above that the limit is in fact 1).
Remark: when applying L’Hôpital’s rule, it is essential to check that the limit “on the right” exists, otherwise applying L’Hôpital’s rule may lead to erroneous conclusions, as shown in the example below.

Example 24

Consider the limit $\lim_{x \to \infty} \frac{x + \sin x}{x}$. It is not difficult to see that

$$\frac{x - 1}{x} \leq \frac{x + \sin x}{x} \leq \frac{x + 1}{x}, \quad x > 0,$$

and therefore

$$1 = \lim_{x \to \infty} \frac{x - 1}{x} \leq \lim_{x \to \infty} \frac{x + \sin x}{x} \leq \lim_{x \to \infty} \frac{x + 1}{x} = 1,$$

which shows that

$$\lim_{x \to \infty} \frac{x + \sin x}{x} = 1.$$

Applying incorrectly L’Hôpital’s rule we obtain

$$\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \frac{(x + \sin x)'}{(x)'} = \lim_{x \to \infty} \frac{1 + \cos x}{1} = \lim_{x \to \infty} (1 + \cos x) - \text{DNE},$$

which “shows” that the given limit does not exist (we proved above that the limit is in fact 1).
**Remark:** when applying L’Hôpital’s rule, it is essential to check that the limit “on the right” exists, otherwise applying L’Hôpital’s rule may lead to erroneous conclusions, as shown in the example below.

### Example 24

Consider the limit \( \lim_{x \to \infty} \frac{x + \sin x}{x} \). It is not difficult to see that

\[
\frac{x - 1}{x} \leq \frac{x + \sin x}{x} \leq \frac{x + 1}{x}, \quad x > 0,
\]

and therefore

\[
1 = \lim_{x \to \infty} \frac{x - 1}{x} \leq \lim_{x \to \infty} \frac{x + \sin x}{x} \leq \lim_{x \to \infty} \frac{x + 1}{x} = 1,
\]

which shows that

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = 1.
\]

**Applying incorrectly** L’Hôpital’s rule we obtain

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \frac{(x + \sin x)'}{(x)'} = \lim_{x \to \infty} \frac{1 + \cos x}{1} = \lim_{x \to \infty} (1 + \cos x) - \text{DNE},
\]

which “shows” that the given limit does not exist (we proved above that the limit is in fact 1).
Remark: when applying L’Hôpital’s rule, it is essential to check that the limit “on the right” exists, otherwise applying L’Hôpital’s rule may lead to erroneous conclusions, as shown in the example below.

Example 24

Consider the limit \( \lim_{x \to \infty} \frac{x + \sin x}{x} \). It is not difficult to see that

\[
\frac{x - 1}{x} \leq \frac{x + \sin x}{x} \leq \frac{x + 1}{x}, \quad x > 0,
\]

and therefore

\[
1 = \lim_{x \to \infty} \frac{x - 1}{x} \leq \lim_{x \to \infty} \frac{x + \sin x}{x} \leq \lim_{x \to \infty} \frac{x + 1}{x} = 1,
\]

which shows that

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = 1.
\]

Applying incorrectly L’Hôpital’s rule we obtain

\[
\lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \frac{(x + \sin x)'}{(x)'} = \lim_{x \to \infty} \frac{1 + \cos x}{1} = \lim_{x \to \infty} (1 + \cos x) - \text{DNE},
\]

which “shows” that the given limit does not exist (we proved above that the limit is in fact 1).
Definition 25 (Second derivative)

Let \( f : A \subset \mathbb{R} \to \mathbb{R} \) and let \( x_0 \in A \). If the derivative \( f' : A \subset \mathbb{R} \to \mathbb{R} \) exists, we define the second derivative of \( f \) at \( x_0 \) by

\[
 f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0},
\]

provided the limit exists.

If the above limit exists and it is finite, we say that the function \( f \) is twice differentiable at \( x_0 \). If \( f \) is twice differentiable at any point \( x_0 \in A \), we say that the function \( f \) is twice differentiable on \( A \) and call the function \( f'' : A \subset \mathbb{R} \to \mathbb{R} \) the second derivative of \( f \).

\(^a\)The point \( x_0 \in A \) should not be an isolated point of \( A \).
Definition 25 (Second derivative)

Let \( f : A \subset \mathbb{R} \to \mathbb{R} \) and let \( x_0 \in A \). If the derivative \( f' : A \subset \mathbb{R} \to \mathbb{R} \) exists, we define the second derivative of \( f \) at \( x_0 \) by

\[
f'' (x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0},
\]

provided the limit exists.

If the above limit exists and it is finite, we say that the function \( f \) is twice differentiable at \( x_0 \).

If \( f \) is twice differentiable at any point \( x_0 \in A \), we say that the function \( f \) is twice differentiable on \( A \) and call the function \( f'' : A \subset \mathbb{R} \to \mathbb{R} \) the second derivative of \( f \).

\[^a\text{The point } x_0 \in A \text{ should not be an isolated point of } A.\]
Higher order derivatives

Definition 25 (Second derivative)

Let \( f : A \subset \mathbb{R} \to \mathbb{R} \) and let \( x_0 \in A^a \). If the derivative \( f' : A \subset \mathbb{R} \to \mathbb{R} \) exists, we define the second derivative of \( f \) at \( x_0 \) by

\[
 f''(x_0) = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{x - x_0},
\]

provided the limit exists.

If the above limit exists and it is finite, we say that the function \( f \) is twice differentiable at \( x_0 \).

If \( f \) is twice differentiable at any point \( x_0 \in A \), we say that the function \( f \) is twice differentiable on \( A \) and call the function \( f'' : A \subset \mathbb{R} \to \mathbb{R} \) the second derivative of \( f \).

\(^a\)The point \( x_0 \in A \) should not be an isolated point of \( A \).
The above definition can be extended inductively, to any $n \in \mathbb{N}$, as follows:

**Definition 26 (Higher order derivatives)**

Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and let $x_0 \in A$ be an accumulation point of $A$. If $f$ is $n$ times differentiable on $A$, we define the $n + 1$ derivative of $f$ at $x_0$ by

$$f^{(n+1)}(x_0) = \lim_{x \to x_0} \frac{f^{(n)}(x) - f^{(n)}(x_0)}{x - x_0}$$

provided the limit exists.

If the above limit exists and it is also finite, we say that the function $f$ is $n + 1$ times differentiable at $x_0$.

If $f$ is $n + 1$ times differentiable at any point $x_0 \in A$, we say that the function $f$ is $n + 1$ times differentiable on $A$ and call the function $f^{(n+1)} : A \subset \mathbb{R} \to \mathbb{R}$ the $n + 1$ derivative of $f$. 
The above definition can be extended inductively, to any $n \in \mathbb{N}$, as follows:

**Definition 26 (Higher order derivatives)**

Let $f : A \subset \mathbb{R} \to \mathbb{R}$ and let $x_0 \in A$ be an accumulation point of $A$. If $f$ is $n$ times differentiable on $A$, we define the $n + 1$ derivative of $f$ at $x_0$ by

$$f^{(n+1)}(x_0) = \lim_{x \to x_0} \frac{f^{(n)}(x) - f^{(n)}(x_0)}{x - x_0}$$

(11)

provided the limit exists.

If the above limit exists and it is also finite, we say that the function $f$ is $n + 1$ times differentiable at $x_0$.

If $f$ is $n + 1$ times differentiable at any point $x_0 \in A$, we say that the function $f$ is $n + 1$ times differentiable on $A$ and call the function $f^{(n+1)} : A \subset \mathbb{R} \to \mathbb{R}$ the $n + 1$ derivative of $f$. 

M. N. Pascu (Transilvania Univ)

Course No. 6 24/30

2.12.2014 24 / 30
The above definition can be extended inductively, to any \( n \in \mathbb{N} \), as follows:

**Definition 26 (Higher order derivatives)**

Let \( f : A \subset \mathbb{R} \rightarrow \mathbb{R} \) and let \( x_0 \in A \) be an accumulation point of \( A \). If \( f \) is \( n \) times differentiable on \( A \), we define the \( n + 1 \) derivative of \( f \) at \( x_0 \) by

\[
f^{(n+1)}(x_0) = \lim_{x \to x_0} \frac{f^{(n)}(x) - f^{(n)}(x_0)}{x - x_0}
\]

(11)

provided the limit exists.

If the above limit exists and it is also finite, we say that the function \( f \) is \( n + 1 \) times differentiable at \( x_0 \).

If \( f \) is \( n + 1 \) times differentiable at any point \( x_0 \in A \), we say that the function \( f \) is \( n + 1 \) times differentiable on \( A \) and call the function \( f^{(n+1)} : A \subset \mathbb{R} \rightarrow \mathbb{R} \) the \( n + 1 \) derivative of \( f \).
Remark 27 (Various notations for derivatives)

- **Lagrange’s prime notation:**

  \[ f(x_0), f'(x_0), f''(x_0), f'''(x_0), \ldots \]

  or

  \[ f(x_0), f^{(1)}(x_0), f^{(2)}(x_0), f^{(3)}(x_0), \ldots \]

- **Leibniz’s differential notation**

  \[ f(x_0), \frac{df}{dx}(x_0), \frac{d^2f}{dx^2}(x_0), \frac{d^3f}{dx^3}(x_0), \ldots \]

- **Newton’s dot notation (mostly used in Physics, for derivatives with respect to time):**

  \[ f(x_0), \dot{f}(x_0), \ddot{f}(x_0), \ldots \]

- **Euler’s notation**

  \[ f(x_0), Df(x_0), D^2f(x_0), D^3f(x_0), \ldots \]
Remark 27 (Various notations for derivatives)

- **Lagrange’s prime notation:**

  \[ f(x_0), f'(x_0), f''(x_0), f'''(x_0), \ldots \]

  or

  \[ f(x_0), f^{(1)}(x_0), f^{(2)}(x_0), f^{(3)}(x_0), \ldots \]

- **Leibniz’s differential notation**

  \[ f(x_0), \frac{df}{dx}(x_0), \frac{d^2f}{dx^2}(x_0), \frac{d^3f}{dx^3}(x_0), \ldots \]

- **Newton’s dot notation (mostly used in Physics, for derivatives with respect to time):**

  \[ f(x_0), \dot{f}(x_0), \ddot{f}(x_0), \dddot{f}(x_0) \ldots \]

- **Euler’s notation**

  \[ f(x_0), Df(x_0), D^2f(x_0), D^3f(x_0), \ldots \]
Remark 27 (Various notations for derivatives)

- **Lagrange’s prime notation:**

\[ f(x_0), f'(x_0), f''(x_0), f'''(x_0), \ldots \]

or

\[ f(x_0), f^{(1)}(x_0), f^{(2)}(x_0), f^{(3)}(x_0), \ldots \]

- **Leibniz’s differential notation**

\[ f(x_0), \frac{df}{dx}(x_0), \frac{d^2f}{dx^2}(x_0), \frac{d^3f}{dx^3}(x_0), \ldots \]

- **Newton’s dot notation** (mostly used in Physics, for derivatives with respect to time):

\[ f(x_0), \dot{f}(x_0), \ddot{f}(x_0), \ldots \]

- **Euler’s notation**

\[ f(x_0), Df(x_0), D^2f(x_0), D^3f(x_0), \ldots \]
If $f : A \subset \mathbb{R} \rightarrow \mathbb{R}$ is differentiable at the point $x_0 \in A$, by definition we have:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \Leftrightarrow \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0 \Leftrightarrow \lim_{x \to x_0} \alpha(x) = 0.$$

So if $f$ is differentiable at $x_0$, then we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \alpha(x)(x - x_0), \quad x \in A,$$

and moreover $\lim_{x \to x_0} \alpha(x) = 0$.

If $f$ is twice differentiable at $x_0$, it can be shown that we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \alpha(x)(x - x_0)^2, \quad x \in A,$$

and moreover $\lim_{x \to x_0} \alpha(x) = 0$.

A similar representation holds if the function $f$ is several times differentiable at $x_0$, leading to the following.
If \( f : A \subset \mathbb{R} \to \mathbb{R} \) is differentiable at the point \( x_0 \in A \), by definition we have:

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \iff \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0 \iff \lim_{x \to x_0} \alpha(x) = 0.
\]

So if \( f \) is differentiable at \( x_0 \), then we have

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \alpha(x)(x - x_0), \quad x \in A,
\]

and moreover \( \lim_{x \to x_0} \alpha(x) = 0 \).

If \( f \) is twice differentiable at \( x_0 \), it can be shown that we have

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \alpha(x)(x - x_0)^2, \quad x \in A,
\]

and moreover \( \lim_{x \to x_0} \alpha(x) = 0 \).

A similar representation holds if the function \( f \) is several times differentiable at \( x_0 \), leading to the following.
If $f : A \subset \mathbb{R} \to \mathbb{R}$ is differentiable at the point $x_0 \in A$, by definition we have:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \iff \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0 \iff \lim_{x \to x_0} \alpha(x) = 0.$$ 

So if $f$ is differentiable at $x_0$, then we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \alpha(x)(x - x_0), \quad x \in A,$$

and moreover $\lim_{x \to x_0} \alpha(x) = 0$.

If $f$ is twice differentiable at $x_0$, it can be shown that we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)\frac{(x - x_0)^2}{2!} + \alpha(x)(x - x_0)^2, \quad x \in A,$$

and moreover $\lim_{x \to x_0} \alpha(x) = 0$.

A similar representation holds if the function $f$ is several times differentiable at $x_0$, leading to the following.
If \( f : A \subset \mathbb{R} \to \mathbb{R} \) is differentiable at the point \( x_0 \in A \), by definition we have:

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \Leftrightarrow \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0 \Leftrightarrow \lim_{x \to x_0} \alpha(x) = 0.
\]

So if \( f \) is differentiable at \( x_0 \), then we have

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \alpha(x)(x - x_0), \quad x \in A,
\]

and moreover \( \lim_{x \to x_0} \alpha(x) = 0 \).

If \( f \) is twice differentiable at \( x_0 \), it can be shown that we have

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \alpha(x)(x - x_0)^2, \quad x \in A,
\]

and moreover \( \lim_{x \to x_0} \alpha(x) = 0 \).

A similar representation holds if the function \( f \) is several times differentiable at \( x_0 \), leading to the following.
If $f : A \subset \mathbb{R} \to \mathbb{R}$ is differentiable at the point $x_0 \in A$, by definition we have:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \iff \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x-x_0)}{x - x_0} = 0 \iff \lim_{x \to x_0} \alpha(x) = 0.$$ 

So if $f$ is differentiable at $x_0$, then we have

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \alpha(x)(x-x_0), \quad x \in A,$$

and moreover $\lim_{x \to x_0} \alpha(x) = 0$.

If $f$ is twice differentiable at $x_0$, it can be shown that we have

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!} (x-x_0)^2 + \alpha(x)(x-x_0)^2, \quad x \in A,$$

and moreover $\lim_{x \to x_0} \alpha(x) = 0$.

A similar representation holds if the function $f$ is several times differentiable at $x_0$, leading to the following.
If \( f : A \subset \mathbb{R} \to \mathbb{R} \) is differentiable at the point \( x_0 \in A \), by definition we have:

\[
\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \iff \lim_{x \to x_0} \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{x - x_0} = 0 \iff \lim_{x \to x_0} \alpha(x) = 0.
\]

So if \( f \) is differentiable at \( x_0 \), then we have

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \alpha(x)(x - x_0), \quad x \in A,
\]

and moreover \( \lim_{x \to x_0} \alpha(x) = 0 \).

If \( f \) is twice differentiable at \( x_0 \), it can be shown that we have

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \alpha(x)(x - x_0)^2, \quad x \in A,
\]

and moreover \( \lim_{x \to x_0} \alpha(x) = 0 \).

A similar representation holds if the function \( f \) is several times differentiable at \( x_0 \), leading to the following.
Theorem 28 (Taylor’s formula)

Let \( f : (a, b) \subset \mathbb{R} \to \mathbb{R} \) be a function which is \( n + 1 \) times differentiable on the interval \((a, b)\) and let \( x_0 \in (a, b)\).

Then for each \( x \in (a, b) \) there exists a point \( \xi \) between \( x_0 \) and \( x \) such that

\[
f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - x_0)^{n+1}
\]

(12)

Proof.

Very clever! The idea is to apply Rolle’s theorem to a well-chosen function - see the notes if you are curious to know how it is done.

Definition 29

Formula (12) above is called Taylor formula of order \( n \) for \( f \) at the point \( x_0 \).

The polynomial \( T_n(x) \) is called the Taylor polynomial of order \( n \) of \( f \) at the point \( x_0 \).

\( R_n(x) \) is called the Taylor remainder of order \( n \) of \( f \) at the point \( x_0 \).
**Theorem 28 (Taylor’s formula)**

Let \( f : (a, b) \subset \mathbb{R} \to \mathbb{R} \) be a function which is \( n + 1 \) times differentiable on the interval \((a, b)\) and let \( x_0 \in (a, b) \).

Then for each \( x \in (a, b) \) there exists a point \( \xi \) between \( x_0 \) and \( x \) such that

\[
f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x-x_0) + \frac{f''(x_0)}{2!} (x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!} (x-x_0)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1}
\]

(12)

**Proof.**

Very clever! The idea is to apply Rolle’s theorem to a well-chosen function - see the notes if you are curious to know how it is done.

**Definition 29**

Formula (12) above is called **Taylor formula of order n** for \( f \) at the point \( x_0 \).

The polynomial \( T_n(x) \) is called the **Taylor polynomial** of order \( n \) of \( f \) at the point \( x_0 \).

\( R_n(x) \) is called the **Taylor remainder** of order \( n \) of \( f \) at the point \( x_0 \).
Theorem 28 (Taylor’s formula)

Let \( f : (a, b) \subset \mathbb{R} \to \mathbb{R} \) be a function which is \( n + 1 \) times differentiable on the interval \((a, b)\) and let \( x_0 \in (a, b) \). Then for each \( x \in (a, b) \) there exists a point \( \xi \) between \( x_0 \) and \( x \) such that

\[
f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - x_0)^{n+1}
\]

(12)

Proof.

Very clever! The idea is to apply Rolle’s theorem to a well-chosen function - see the notes if you are curious to know how it is done.

Definition 29

Formula (12) above is called \textbf{Taylor formula of order} \( n \) \textbf{for} \( f \) \textbf{at the point} \( x_0 \).

The polynomial \( T_n(x) \) is called the \textbf{Taylor polynomial} of order \( n \) of \( f \) at the point \( x_0 \).

\( R_n(x) \) is called the \textbf{Taylor remainder} of order \( n \) of \( f \) at the point \( x_0 \).
**Theorem 28 (Taylor’s formula)**

Let \( f : (a, b) \subset \mathbb{R} \to \mathbb{R} \) be a function which is \( n + 1 \) times differentiable on the interval \((a, b)\) and let \( x_0 \in (a, b) \).

Then for each \( x \in (a, b) \) there exists a point \( \xi \) between \( x_0 \) and \( x \) such that

\[
f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - x_0)^{n+1}
\]

Proof.

**Very clever!** The idea is to apply Rolle’s theorem to a well-chosen function - see the notes if you are curious to know how it is done.

Definition 29

Formula (12) above is called **Taylor formula of order \( n \) for \( f \) at the point \( x_0 \).**

The polynomial \( T_n(x) \) is called the **Taylor polynomial** of order \( n \) of \( f \) at the point \( x_0 \).

\( R_n(x) \) is called the **Taylor remainder** of order \( n \) of \( f \) at the point \( x_0 \).
Theorem 28 (Taylor’s formula)

Let \( f : (a, b) \subset \mathbb{R} \to \mathbb{R} \) be a function which is \( n + 1 \) times differentiable on the interval \((a, b)\) and let \( x_0 \in (a, b) \).

Then for each \( x \in (a, b) \) there exists a point \( \xi \) between \( x_0 \) and \( x \) such that

\[
f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - x_0)^{n+1}
\]

Definition 29

Formula (12) above is called **Taylor formula of order** \( n \) **for** \( f \) **at the point** \( x_0 \).

The polynomial \( T_n(x) \) is called the **Taylor polynomial** of order \( n \) of \( f \) at the point \( x_0 \).

\( R_n(x) \) is called the **Taylor remainder** of order \( n \) of \( f \) at the point \( x_0 \).

Proof.

**Very clever!** The idea is to apply Rolle’s theorem to a well-chosen function - see the notes if you are curious to know how it is done.
Remark 30

The formula (12) in the previous theorem can be written in the form

\[ f(x) = T_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - x_0)^{n+1}. \]

Using the substitution \( x = x_0 + h \), the above formula can be written in the equivalent form:

\[ f(x_0 + h) = T_n(x_0 + h) + R_n(x_0 + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + \frac{f^{(n+1)}(\xi)}{(n + 1)!} h^{n+1}. \]

Since \( R_n(x) \approx 0 \) for \( x \approx x_0 \), Taylor’s formula shows that near \( x_0 \) we can approximate the function \( f \) by the corresponding Taylor polynomial \( T_n(x) \):

\[ f(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k. \]
The formula (12) in the previous theorem can be written in the form

\[ f(x) = T_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - x_0)^{n+1}. \]

Using the substitution \( x = x_0 + h \), the above formula can be written in the equivalent form:

\[ f(x_0 + h) = T_n(x_0 + h) + R_n(x_0 + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + \frac{f^{(n+1)}(\xi)}{(n + 1)!} h^{n+1}. \]

Since \( R_n(x) \approx 0 \) for \( x \approx x_0 \), Taylor’s formula shows that near \( x_0 \) we can approximate the function \( f \) by the corresponding Taylor polynomial \( T_n(x) \):

\[ f(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k. \]
Remark 30

The formula (12) in the previous theorem can be written in the form

\[
 f(x) = T_n(x) + R_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}.
\]

Using the substitution \( x = x_0 + h \), the above formula can be written in the equivalent form:

\[
 f(x_0 + h) = T_n(x_0 + h) + R_n(x_0 + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}.
\]

Since \( R_n(x) \approx 0 \) for \( x \approx x_0 \), Taylor’s formula shows that near \( x_0 \) we can approximate the function \( f \) by the corresponding Taylor polynomial \( T_n(x) \):

\[
 f(x) \approx T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.
\]
Theorem 31 (Sufficient conditions for extremum of a function)

Let \( f : (a, b) \subset \mathbb{R} \to \mathbb{R} \) be a function which is \( n + 1 \geq 2 \) times differentiable with \( f^{(n+1)}(x) \) continuous on the interval \( (a, b) \), \( x_0 \in (a, b) \) and assume that

\[
f'(x_0) = f''(x_0) = \ldots = f^{(n)}(x_0) = 0 \quad \text{and} \quad f^{(n+1)}(x_0) \neq 0.
\]

Then if:

a) \( n + 1 \) is an even number, \( f \) has a local extremum at \( x_0 \). Moreover, if
   i) \( f^{(n+1)}(x_0) > 0 \) then \( x_0 \) is a local minimum point for \( f \);
   ii) \( f^{(n+1)}(x_0) < 0 \) then \( x_0 \) is a local maximum point for \( f \);

b) \( n + 1 \) is an odd number, \( f \) does not have a local extremum at \( x_0 \).

Proof.

Since first \( n \) derivatives of \( f \) are zero at \( x_0 \), Taylor formula of order \( n \) for \( f \) at \( x_0 \) becomes in this case

\[
f(x) - f(x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-x_0)^{n+1}.
\]

If \( n + 1 \) is an even number, \( f(x) - f(x_0) \) has the same sign as \( f^{(n+1)}(\xi) \), so \( x_0 \) is an extremum for \( f \).
If \( n + 1 \) is odd, \( x_0 \) is not an extremum for \( f \) (\( f(x) - f(x_0) \) is both positive and negative for \( x \) near \( x_0 \)).
Theorem 31 (Sufficient conditions for extremum of a function)

Let \( f : (a, b) \subset \mathbb{R} \to \mathbb{R} \) be a function which is \( n + 1 \geq 2 \) times differentiable with \( f^{(n+1)} (x) \) continuous on the interval \((a, b)\), \( x_0 \in (a, b) \) and assume that

\[
f' (x_0) = f'' (x_0) = \ldots = f^{(n)} (x_0) = 0 \quad \text{and} \quad f^{(n+1)} (x_0) \neq 0.
\]

Then if:

a) \( n + 1 \) is an even number, \( f \) has a local extremum at \( x_0 \). Moreover, if
   i) \( f^{(n+1)} (x_0) > 0 \) then \( x_0 \) is a local minimum point for \( f \);
   ii) \( f^{(n+1)} (x_0) < 0 \) then \( x_0 \) is a local maximum point for \( f \);

b) \( n + 1 \) is an odd number, \( f \) does not have a local extremum at \( x_0 \).

Proof.

Since first \( n \) derivatives of \( f \) are zero at \( x_0 \), Taylor formula of order \( n \) for \( f \) at \( x_0 \) becomes in this case

\[
f (x) - f (x_0) = \frac{f^{(n+1)} (\xi)}{(n + 1)!} (x - x_0)^{n+1}.
\]

If \( n + 1 \) is an even number, \( f (x) - f (x_0) \) has the same sign as \( f^{(n+1)} (\xi) \), so \( x_0 \) is an extremum for \( f \).
If \( n + 1 \) is odd, \( x_0 \) is not an extremum for \( f \) (\( f (x) - f (x_0) \) is both positive and negative for \( x \) near \( x_0 \)).
Theorem 31 (Sufficient conditions for extremum of a function)

Let \( f : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R} \) be a function which is \( n + 1 \geq 2 \) times differentiable with \( f^{(n+1)}(x) \) continuous on the interval \((a, b)\), \( x_0 \in (a, b) \) and assume that

\[
\begin{align*}
    f'(x_0) &= f''(x_0) = \ldots = f^{(n)}(x_0) = 0 \quad \text{and} \quad f^{(n+1)}(x_0) \neq 0.
\end{align*}
\]

Then if:

a) \( n + 1 \) is an even number, \( f \) has a local extremum at \( x_0 \). Moreover, if
   i) \( f^{(n+1)}(x_0) > 0 \) then \( x_0 \) is a local minimum point for \( f \);
   ii) \( f^{(n+1)}(x_0) < 0 \) then \( x_0 \) is a local maximum point for \( f \);

b) \( n + 1 \) is an odd number, \( f \) does not have a local extremum at \( x_0 \).

Proof.

Since first \( n \) derivatives of \( f \) are zero at \( x_0 \), Taylor formula of order \( n \) for \( f \) at \( x_0 \) becomes in this case

\[
f(x) - f(x_0) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - x_0)^{n+1}.
\]

If \( n + 1 \) is an even number, \( f(x) - f(x_0) \) has the same sign as \( f^{(n+1)}(\xi) \), so \( x_0 \) is an extremum for \( f \).

If \( n + 1 \) is odd, \( x_0 \) is not an extremum for \( f \) (\( f(x) - f(x_0) \) is both positive and negative for \( x \) near \( x_0 \)).

\( \square \)
Theorem 31 (Sufficient conditions for extremum of a function)

Let \( f : (a, b) \subset \mathbb{R} \to \mathbb{R} \) be a function which is \( n + 1 \geq 2 \) times differentiable with \( f^{(n+1)}(x) \) continuous on the interval \((a, b)\), \( x_0 \in (a, b) \) and assume that

\[
f'(x_0) = f''(x_0) = \ldots = f^{(n)}(x_0) = 0 \quad \text{and} \quad f^{(n+1)}(x_0) \neq 0.
\]

Then if:

a) \( n + 1 \) is an even number, \( f \) has a local extremum at \( x_0 \). Moreover, if
   i) \( f^{(n+1)}(x_0) > 0 \) then \( x_0 \) is a local minimum point for \( f \);
   ii) \( f^{(n+1)}(x_0) < 0 \) then \( x_0 \) is a local maximum point for \( f \);

b) \( n + 1 \) is an odd number, \( f \) does not have a local extremum at \( x_0 \).

Proof.

Since first \( n \) derivatives of \( f \) are zero at \( x_0 \), Taylor formula of order \( n \) for \( f \) at \( x_0 \) becomes in this case

\[
f(x) - f(x_0) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - x_0)^{n+1}.
\]

If \( n + 1 \) is an even number, \( f(x) - f(x_0) \) has the same sign as \( f^{(n+1)}(\xi) \), so \( x_0 \) is an extremum for \( f \).
If \( n + 1 \) is odd, \( x_0 \) is not an extremum for \( f \) (\( f(x) - f(x_0) \) is both positive and negative for \( x \) near \( x_0 \)).
Theorem 31 (Sufficient conditions for extremum of a function)

Let \( f : (a, b) \subset \mathbb{R} \rightarrow \mathbb{R} \) be a function which is \( n + 1 \geq 2 \) times differentiable with \( f^{(n+1)} (x) \) continuous on the interval \( (a, b) \), \( x_0 \in (a, b) \) and assume that

\[
f' (x_0) = f'' (x_0) = \ldots = f^{(n)} (x_0) = 0 \quad \text{and} \quad f^{(n+1)} (x_0) \neq 0.
\]

Then if:

a) \( n + 1 \) is an even number, \( f \) has a local extremum at \( x_0 \). Moreover, if
   i) \( f^{(n+1)} (x_0) > 0 \) then \( x_0 \) is a local minimum point for \( f \);
   ii) \( f^{(n+1)} (x_0) < 0 \) then \( x_0 \) is a local maximum point for \( f \);

b) \( n + 1 \) is an odd number, \( f \) does not have a local extremum at \( x_0 \).

Proof.

Since first \( n \) derivatives of \( f \) are zero at \( x_0 \), Taylor formula of order \( n \) for \( f \) at \( x_0 \) becomes in this case

\[
f (x) - f (x_0) = \frac{f^{(n+1)} (\xi)}{(n + 1)!} (x - x_0)^{n+1}.
\]

If \( n + 1 \) is an even number, \( f (x) - f (x_0) \) has the same sign as \( f^{(n+1)} (\xi) \), so \( x_0 \) is an extremum for \( f \). If \( n + 1 \) is odd, \( x_0 \) is not an extremum for \( f \) (\( f (x) - f (x_0) \) is both positive and negative for \( x \) near \( x_0 \)).
Example 32

Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by $f (x) = \sin^2 x$ and $g (x) = x \sin^2 x$.

It is easy to see that

\[
f'(0) = 0 \quad \text{and} \quad f''(0) = 2 \neq 0,
\]

and

\[
g'(0) = g''(0) = 0 \quad \text{and} \quad g'''(0) = 6 \neq 0.
\]

By the above theorem it follows that:

- $x_0 = 0$ is a relative minimum point for $f$ (note that $n + 1 = 2$ is an even number and $f'''(0) > 0$);
- $x_0$ is not a relative extremum point for $g$ (note that $n + 1 = 3$ is an odd number).
Example 32

Consider the functions \( f, g : \mathbb{R} \to \mathbb{R} \) given by \( f(x) = \sin^2 x \) and \( g(x) = x \sin^2 x \).

It is easy to see that
\[
 f'(0) = 0 \quad \text{and} \quad f''(0) = 2 \neq 0,
\]
and
\[
 g'(0) = g''(0) = 0 \quad \text{and} \quad g'''(0) = 6 \neq 0.
\]

By the above theorem it follows that:

- \( x_0 = 0 \) is a relative minimum point for \( f \) (note that \( n + 1 = 2 \) is an even number and \( f''(0) > 0 \));
- \( x_0 \) is not a relative extremum point for \( g \) (note that \( n + 1 = 3 \) is an odd number).
Example 32

Consider the functions \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) given by \( f(x) = \sin^2 x \) and \( g(x) = x \sin^2 x \).

It is easy to see that

\[
    f'(0) = 0 \text{ and } f''(0) = 2 \neq 0,
\]

and

\[
    g'(0) = g''(0) = 0 \text{ and } g'''(0) = 6 \neq 0.
\]

By the above theorem it follows that:

- \( x_0 = 0 \) is a relative minimum point for \( f \) (note that \( n + 1 = 2 \) is an even number and \( f'''(0) > 0 \));
- \( x_0 \) is not a relative extremum point for \( g \) (note that \( n + 1 = 3 \) is an odd number).
Example 32

Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin^2 x$ and $g(x) = x \sin^2 x$.

It is easy to see that

$$f'(0) = 0 \quad \text{and} \quad f''(0) = 2 \neq 0,$$

and

$$g'(0) = g''(0) = 0 \quad \text{and} \quad g'''(0) = 6 \neq 0.$$

By the above theorem it follows that:

- $x_0 = 0$ is a relative minimum point for $f$ (note that $n + 1 = 2$ is an even number and $f'''(0) > 0$);
- $x_0$ is not a relative extremum point for $g$ (note that $n + 1 = 3$ is an odd number).